Math 5220–Complex Analysis: Problem Set 3

Gill

Due: Friday February 9, 2018

Problem 1

The series $\sum_{n=1}^{\infty} n^2 z^n$ and $\sum_{n=1}^{\infty} n^3 z^n$ converge for $z \in \mathbb{D}(0,1)$. Find their sums.

Problem 2

Expand $\frac{2z-1}{z+1}$ and $\frac{1}{z+1}^2$ in powers of $z-1$. Express you answers in the form $\sum_{n=1}^{\infty} b_n (z-1)^n$, and write down formulae for the b_n. Also, find the region of convergence of the series, and sketch it.

Problem 3

Find the radius of convergence of the following power series.

$$\text{(a)} \sum_{n=0}^{\infty} 2^n z^{3n} \qquad \text{(b)} \sum_{n=0}^{\infty} 2^n z^{3^n} \qquad \text{(c)} \sum_{n \text{ odd}, n \geq 1} (1 - \frac{1}{n})^{n^2} z^n$$

Problem 4

Let $\{a_n\}$ be a sequence in $\mathbb{R}^+$ with $a_n \searrow 0$ as $n \nearrow \infty$. Set $f(z) = \sum_{n=1}^{\infty} a_n z^n$. The radius of convergene of the series is at least 1. Set also $K(\delta) = \overline{\mathbb{D}}(0,1) \setminus \mathbb{D}(1,\delta)$. Prove that the series converges uniformly on $K(\delta)$, for each $\delta > 0$.

In particular, the series $\sum_{n=1}^{\infty} a_n z^n$ converges for each $e^{i\theta}$ on the unit circle, except perhaps for $e^{i\theta} = 1$. A prototypical example is $\sum_{n=1}^{\infty} 1/ne^{in\theta}$, which diverges for $e^{i\theta} = 1$. For $e^{i\theta} \neq 1$, it turns out that the sum of this series is $\log \frac{1}{1-e^{i\theta}}$, where $\log \frac{1}{1-z}$ denotes the branch of the logarithm which is holomorphic in $\mathbb{C} \setminus [1, \infty)$ and equal to zero at $z = 0$.

To prove the assertion, fix $\delta > 0$. Show that the partial sums of the series form a uniform Cauchy sequence on $K(\delta)$. To acheive this, use epsilons, m's, n's, and summation by parts.

Problem 5

The Fibonacci sequence is defined by the recurrence relation

$$c_0 = c_1 = 1, \qquad c_n = c_{n-1} + c_{n-2}, \qquad n \geq 2.$$

(a) Prove that $|c_n| \leq 2^n$ for all $n \geq 0$. Thus $\sum_{n=0}^{\infty} c_n z^n$ converges at least in the disk $\mathbb{D}(0, 1/2)$ to a function f.

(b) Find f (it's rational).

(c) By decomposing f into partial fractions, determine the exact value of the radius of convergence of the series, and a formula for the c_n.

Problem 6 Do not turn in, for your own edification.

Find the radius of convergence for each of the following power series.

$$\text{(a)} \sum_{n=1}^{\infty} (n!)^{n^{-1/2}} z^n \qquad \text{(b)} \sum_{n=2}^{\infty} (n!)^{(log n)^{-1/2}} z^n \qquad \text{(c)} \sum_{n=1}^{\infty} \frac{n^{\alpha}}{n!} z^n \qquad \text{(d)} \sum_{n=1}^{\infty} \frac{n^n}{n!} z^n.$$

In (c), α is a fixed positive number. You may use Stirling's formula:

$$n! \sim n^n e^{-n} (2\pi n)^{1/2}, \qquad n \to \infty$$

where $a_n \sim b_n$ measns that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.