Math 522–Complex Analysis: Problem Set 2

Gill

Due: Friday, February 2

Problem 1

Give an example of a function $f = u + iv : \mathbb{C} \to \mathbb{C}$ with the following properties:

1. $f \in C^\infty(\mathbb{C} \setminus \{0\})$
2. f is continuous at 0.
3. The partial derivatives u_x, u_y, v_x, and v_y exist at 0, at least one is non-zero at 0, $u_x(0) = v_y(0)$, and $v_x(0) = -u_y(0)$.
4. $f'(0)$ does not exist.

Problem 2

The inverse stereographic projection T maps circles $K \subset S^2$ to circles in $\mathbb{C}$ when K does not contain the North Pole, and maps circles through the North Pole to sets of the form $L \cup \{\infty\}$, where L is a line in $\mathbb{C}$. Illustrate this fact in each of the following cases.
(a) K is the circle on S^2 with center $(0, 0, a)$ and radius $(1 - a^2)^{1/2}$, where $a \in (-1, 1)$.
(b) L is the line $y = \lambda(x - 1)$, where $\lambda \in \mathbb{R}$.
In (a), you should determine the center and radius of $T(K)$ in $\mathbb{C}$, and show that points on $T(K)$ satisfy an appropriate equation. In (b), you should find the center and radius of $T^{-1}(L)$ in $\mathbb{R}^3$. Try to draw this circle in a sketch of S^2.

Problem 3

(a) Write $P(x, y) = (x^2y) + 2ixy$ as a polynomial in z and $\overline{z}$.
(b) Calculate each of P_z and $P_{\overline{z}}$ in two different ways, and make sure the answers which should be the same are really the same.

Problem 4

Let $D = \mathbb{C} \setminus [\{-\frac{1}{n^2} : n \in \mathbb{Z}^+\} \cup \{0\}]$. Use the Weierstrass M-test to prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{1 + zn^2}$$

converges uniformly on each compact subset of D.

Problem 5

Let $f(z) = az + b\overline{z}$, where a and b are complex constants, at least one of which is not zero.

(a) Suppose that $|a| > |b|$. Show that f maps the positively oriented unit circle $\partial\mathbb{D}$ onto a positively oriented ellipse with center at the origin. Determine the length of the major and minor semiaxes, and the angle $\phi \in [0, \pi)$ which the major semiaxis makes with the positive real axes. Draw a sketch.

(b) Describe the image of $\partial\mathbb{D}$ when $|a| = |b|$.

Hint: Here is a good outline to use for your problem. You may assume that $e^{i\theta} = \cos\theta + i\sin\theta$ and the polar representation of complex numbers as this will be shown shortly.

(1) Suppose that a and b are both positive real. Then F maps the unit circle $\partial\mathbb{D}$ onto an ellipse with major axis along the x-axis. Find the equation of the ellipse. What are the length of the major and minor semi-axes?

(2) Write $a = Ae^{i\phi_1}$, $b = Be^{i\phi_2}$, where A and B are positive and the ϕ_j are real. Find real ψ and χ such that

$$F(z) = e^{i\chi}F_0(ze^{-i\psi}),$$

where F_0 has the form of the F in (1).

(3) Conclude that F maps the unit circle $1-1$ onto an ellipse centered at the origin. Find, in terms of a and b, then lengths of the major and mino semiaxes, the ange in $[0, \pi)$ which the major semiaxis makes with the positive real axis, and the points of $\partial\mathbb{D}$ at which $|F|$ is largest and smallest.