Math 5220–Complex Analysis: Problem Set 11

Gill

Due: Friday May 4, 2018

Problem 1

Prove the following principle for harmonic functions: If u is nonconstant and harmonic in a domain D, then u has neither a local maximum nor a local minimum at any point of D.

Problem 2

Give an example of a bounded harmonic function u in $\mathbb{D}$ whose conjugate harmonic function v is not bounded in $\mathbb{D}$.

Problem 3

Let $f = u + iv \in H(\mathbb{D})$, with $f(0) = 0$ and $\sup_{\mathbb{D}} |u| \leq 1$. For each $z_0 \in \mathbb{D}$, find sharp upper bounds for $|v(z_0)|$ and $|f(z_0)|$.

Problem 4

Let $f = u + iv$ be an entire function with $f(0) = 0$. Find an absolute constant C such that

$$M(r, f) \leq C \sup_{|z|=2r} |u(z)|, \qquad 0 < r < \infty.$$

Deduce that if $u(z) = O(|z|^\alpha)$ as $z \to \infty$ for some $\alpha > 0$, then f is a polynomial.

Problem 5

Suppose that $\mathcal{F}$ is a normal family in $H(D)$. Let $\mathcal{F}_n = \{f^{(n)} : f \in \mathcal{F}\}$, $n \geq 1$. (a) Prove that each $\mathcal{F}_n$ is normal. (b) Prove or disprove: $\cup_{n=1}^{\infty} \mathcal{F}_n$ is always normal.

Problem 6

Suppose that $\phi : \{0, 1, 2, \ldots\} \to [0, \infty)$ satisfies

$$\limsup_{n\to\infty} \phi(n)^{1/n} \leq 1.$$

Let $\mathcal{F} = \mathcal{F}(\phi)$ denote the set of all functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $H(\mathbb{D})$ such that $|a_n| \leq \phi(n)$ for all $n \geq 0$.

(a) Prove that $\mathcal{F}$ is normal in $\mathbb{D}$. This shows in particular that the set of all f satisfying conditions like $|a_n| \leq n^3$, for all $n \geq 0$, is normal.

(b) Does a converse hold? That is, for each normal family $\mathcal{F} \subset H(\mathbb{D})$ does there exist ϕ satsifying the above such that $\mathcal{F} \subset \mathcal{F}(\phi)$