Math 5220–Complex Analysis: Problem Set 10

Gill

Due: Friday April 27, 2018

Problem 1

In parts (a) and (b) a class of holomorphic functions is given. In each case find the largest possible value of the given quantities, and identify the extremal functions. The set Ω denotes the right half plane: $\Omega = \{z = x + iy \in \mathbb{C} : x > 0\}$.

(a) Maximize $|f'(0)|$ and $M(r, f)$ over $\mathcal{A} = \{f \in H(\mathbb{D} : f(\mathbb{D}) \subset \Omega, f(0) = 1\}$. Here $0 < r < 1$.

(b) Maximize $|f'(1)|$ an $|f(x)|$ over $\mathcal{B} = \{f \in H(\Omega) : f(\Omega) \subset \mathbb{D}, f(1) = 0\}$. Here $0 < x < \infty$.

Problem 2

Prove the *Schwarz-Pick lemma*: If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then for all $z \in \mathbb{D}$,

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

Discuss the case of equality.

The Schwarz-Pick lemma may be written as

$$\frac{|dw|}{1 - |w|^2} \leq \frac{|dz|}{1 - |z|^2}.$$

This says that holomorphic mappings are contractions in the hyperbolic metric.

Problem 3

Prove that distinct points z_1, z_2, z_3, z_4 lie on the same line or circle if and only if their cross ratio is real.

Problem 4

Let A be the annulus $\{z \in \mathbb{C} : R_1 < |z| < R_2\}$ where $0 < R_1 < R_2 < \infty$. Prove that there exists no function g holomoprhic in A such that $g^2(z) = z$ in A.

Problem 5

Suppose that f is holomorphic in a neighborhood of z_0, that $w_0 = f(z_0)$, and that $f'(z_0) \neq 0$. Then f injectively maps small circles $|z - z_0| = R$ onto Jordan curves which are almost small circles $|w - w_0| = |f'(z_0)|R$. State a prove a precise version of this phenomemon.

Problem 6 Do not turn in

True or false? There exists an entire function f such that $f(1/n) = e^{|1/n|}$ for each $n \in \mathbb{Z} \setminus \{0\}$.