Math 5220–Complex Analysis: Problem Set 1

Gill

Due: Friday, January 26

Problem 1

A function $f : S \to \mathbb{C}$ is said to be bounded from below if there exists a constant $M > 0$, such that $|f(z)| \geq M$ for every $z \in S$. Prove, using the definition of limit and of bounded from below, that if f and g are functions defined in a neighborhood of $z_0 \in \mathbb{C}$, except perhaps at z_0, if $\lim_{z \to z_0} f(z) = \infty$, and if g is bounded from below, then $\lim_{z \to z_0} f(z)g(z) = \infty$.

Problem 2

Prove: If P is a nonconstant polynomial, then $\lim_{z \to \infty} P(z) = \infty$. You may *not* use the Fundemental Theorem of Algebra, nor the result that a polynomial can be written as a product of linear factors.

Problem 3

Suppose that $f = u + iv$ and that $f'(z_0)$ exists. Show that:
(a) $u_x(z_0) = v_y(z_0)$ and $u_y(z_0) = -v_x(z_0)$
(b)$|f'(z_0)|^2 = (u_x(z_0))^2 + (v_x(z_0))^2 = (u_x(z_0))^2 + (u_y(z_0))^2$.
Suggestions: Use the connections among f', f_x, f_y we found in class, and the facts that $f_x = u_x + iv_x$, $f_y = u_y + iv_y$, provide the decompositions of f_x and f_y into real and imaginary parts.

Problem 4

Suppose that $f = u + iv$ is defined in a neighborhood of $z_0 \in \mathbb{C}$. Prove: If $f'(z_0)$ and $(\overline{f})'(z_0)$ both exist, then $f'(z_0) = 0$.

Problem 5

Suppose that $z = T(x_1, x_2, x_3)$, where $T : S^2 \to \mathbb{C}^*$ is the inverse stereographic projection.
(a) Using, if you like, the formula

$$z = \frac{x_1 + ix_2}{1 - x_3}, \qquad (x_1, x_2, x_3) \in S^2 \setminus \{(0, 0, 1)\},$$

express x_3 as a function of $r = |z|$, and express r as a function of x_3.
(b) Verify that $\chi(\infty, z) = \frac{2}{(1+|z|^2)^{1/2}}$ and that $\chi(0, z) = !*$, where χ is the chordal metric on $\mathbb{C}^*$ and $!*$ is an expression you need to supply.

Problem 6 Do not turn this problem in

Find the real and imaginary parts, and the absolute value, of $z = \frac{1}{(3+i)(1+4i)}$.