Sometimes when mathematicians prove a theorem, they find it easier and more illuminating to prove something more general than what they are interested in. For an example of this from calculus, consider the problem of showing that the number e is given by the following series:

\[ e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots \]

This can be a very difficult problem. One way to show this, is to solve a more general problem. That is, show that \( e^x \) is given by the following Maclaurin series:

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \]

Do you see how the two problems are related? It is possible to find a series expansion for \( \pi \) in a similar way. These series expansions are one way in which e and \( \pi \) can be calculated to many decimal places.

In this class, we are only interested in integers. However, it may be helpful to prove a lemma about all real numbers, then apply it to some integer. The following lemma can be shown by polynomial long division, or it can be proved in other ways as well.

**Lemma 1.** For any natural number \( m \), \((x - 1)\) is a factor of the polynomial \((x^m - 1)\).

It may be useful to apply Lemma 1 at certain integers to prove the following theorem.

**Theorem 1.** If \( n \) is a natural number and \( 2^n - 1 \) is prime, then \( n \) must be prime.

Primes of the form \( 2^n - 1 \) are called Mersenne primes. The largest known prime number is the Mersenne prime \( 2^{43,112,609} - 1 \). Primes of the form \( 2^{2^k} + 1 \) are called Fermat primes. It may be useful to prove a lemma about factors of polynomials to prove the following theorem.

**Theorem 2.** If \( n \) is a natural number and \( 2^n + 1 \) is prime, then \( n \) must be a power of 2.