Math266 Worksheet on 4/22/2013

Given a set $A$, recall that $\mathcal{P}(A)$ is the set of all subsets of $A$.

Example 1. Let $A = \{1, 2, 3\}$. Write out the elements of $\mathcal{P}(A)$. (There should be 8 elements of $\mathcal{P}(A)$.)

Example 2. Let $A = \{1, 2, 3\}$. Define a function $g : A \to \mathcal{P}(A)$ by

$$g(1) = \emptyset, \ g(2) = \{2, 3\}, \ g(3) = \{1, 2\}.$$

Define a subset $B \subseteq A$ by

$$B = \{x \in A : x \notin g(x)\}.$$ 

List the elements of $B$. Does there exist $x \in A$ such that $g(x) = B$?

Example 3. Let $A = \{1, 2, 3, 4, 5\}$. Define a function $g : A \to \mathcal{P}(A)$ by

$$g(1) = \{3\}, \ g(2) = \{2, 4\}, \ g(3) = \{1, 2, 3, 4, 5\}, \ g(4) = \emptyset, \ g(5) = \{1, 2, 4\}.$$

Define a subset $B \subseteq A$ by

$$B = \{x \in A : x \notin g(x)\}.$$ 

List the elements of $B$. Does there exist $x \in A$ such that $g(x) = B$?
Fill in the gaps of the proof

**Theorem 1.** If $A$ is any non-empty set then $|A| < |\mathcal{P}(A)|$

**Proof.** Let $A$ be a non-empty set. We first prove that $|A| \leq |\mathcal{P}(A)|$ by giving a one-to-one function $f : A \rightarrow \mathcal{P}(A)$. For $a \in A$, we let $f(a) = \{a\}$. The function $f$ is one-to-one because ______________. Thus, $|A| \leq |\mathcal{P}(A)|$.

We now show that $|A| \neq |\mathcal{P}(A)|$. For the sake of contradict we assume that $|A| = |\mathcal{P}(A)|$. Thus, there exists a bijection $g : A \rightarrow \mathcal{P}(A)$. We define a subset $B \subseteq A$ by

$$B = \{x \in A : x \notin g(x)\}.$$  

As $g$ is a bijection, there exists $y \in A$ such that $g(y) = B$. We have that $y \in g(y)$ or $y \notin g(y)$. For the case that $y \in g(y)$, we have that __________ by the definition of $B$. This is a contradiction because $g(y) = B$. For the case that $y \notin g(y)$, we have that __________ by the definition of $B$. This is a contradiction because $g(y) = B$. Both cases lead to a contradiction, and hence our assumption that $|A| = |\mathcal{P}(A)|$ is false.

We have that $|A| \leq |\mathcal{P}(A)|$ and $|A| \neq |\mathcal{P}(A)|$. Thus, __________. \qed