Explain why each of the following proofs is incorrect, and then correct the mistakes.

**Theorem 1.** \( \forall n \in \mathbb{N}, \)
\[
\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}
\]

**Proof.** \( \sum_{i=1}^{1} i^2 = 1 = \frac{1(1+1)(2(1)+1)}{6} \). Thus, \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \) for \( n = 1 \).

Let \( k \in \mathbb{N} \), and assume that \( \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6} \). We will show that,
\[
\sum_{i=1}^{k+1} i^2 = \frac{(k + 1)(k + 2)(2(k + 1) + 1)}{6} = \frac{(k + 1)(2k^2 + 7k + 6)}{6} = \frac{(k + 1)(2k^2 + k + 6k + 6)}{6} = \frac{(k + 1)[k(2k + 1) + 6(k + 1)]}{6} = \frac{(k + 1)k(2k + 1)}{6} + (k + 1)^2 = \sum_{i=1}^{k} i^2 + (k + 1)^2 \quad \text{by substituting the induction hypothesis}
\]
\[
\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k + 1)^2
\]

Thus, \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \) for all \( n \in \mathbb{N} \) by induction. \( \square \)
Theorem 2. For every non-negative integer $n$,

$$3|(2^{2n} - 1)$$

Proof. $2^{2(1)} - 1 = 3$. Thus, $3|(2^{2n} - 1)$ for $n = 1$.

Let $k \in \mathbb{Z}$ such that $k \geq 1$ and assume that $3|(2^{2k} - 1)$. We need to show that $3|(2^{2(k+1)} - 1)$.

$$2^{2(k+1)} - 1 = 2^{2k+2} - 1$$
$$= 4(2^{2k}) - 1$$
$$= 4(2^{2k} - 1 + 1) - 1$$
$$= 4(3z + 1) - 1 \quad \text{because } 3|(2^{2k} - 1)$$
$$= 12z + 3$$
$$= 3(4z + 1)$$

Thus, $3|(2^{2(k+1)} - 1)$. This proves that $3|(2^{2n} - 1)$ for all $n \in \mathbb{N}$ by induction. 

\[\square\]