1. Chapter 4 review

1) Prove using the definition of even: For all integers $n$, if $n$ is even then $(-1)^n = 1$.

Let $n \in \mathbb{Z}$ be an even number.

$\exists k \in \mathbb{Z}$ such that $n = 2k$

$(-1)^n = (-1)^{2k} = (\overline{-1})^k = 1^k = 1$

Thus, $(-1)^n = 1$

2) Prove using the definition of odd: The product of any two odd integers is odd.

Let $n, m \in \mathbb{Z}$ be odd.

$\exists k \in \mathbb{Z}$ such that $n = 2k+1$

$\exists k \in \mathbb{Z}$ such that $m = 2k+1$

$n \cdot m = (2k+1)(2k+1) = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$

Thus, $n \cdot m$ is odd.
3) Prove: For each integer \( n \) with \( 1 \leq n \leq 5 \), \( n^2 - n + 11 \) is prime.

4) Show that: \(.123123123...\) is a rational number.

5) Prove using the definition of divides: For all integers \( a, b, \) and \( c \), if \( a \) divides \( b \) and \( b \) divides \( c \) then \( a \) divides \( c \).

Let \( a, b, c \in \mathbb{Z} \)

Assume that \( a \) divides \( b \) and \( b \) divides \( c \).

There exist \( n, m \in \mathbb{Z} \) such that
\[
b = an \quad \text{and} \quad c = bm
\]

Thus, \( c = (an)m \) by substitution
\[
= a(n \cdot m)
\]

Hence, \( a \) divides \( c \).
6) Evaluate 50 \text{ div } 4 \text{ and } 50 \text{ mod } 4.

7) Prove using the definition of \text{ mod}: For every integer \( p \), if \( p \mod 5 = 2 \) then \( 4p \mod 5 = 3 \).

This is an example. We need to prove it for all \( p \in \mathbb{Z} \).

\[ 2 \mod 5 = 2 \]
\[ (4 \cdot 2) \mod 5 = 8 \mod 5 = 3 \]

8) Prove: For all integers \( n \), \( n^2 - n \) is even.
9 a) How do you prove a statement by contradiction?

9 b) Prove: There is no greatest negative rational number.

Assume that there is a greatest negative rational number. Let $x$ be the greatest negative rational.

$\exists \ p, q \in \mathbb{Z}$ such that $q \neq 0$ and $x = \frac{p}{q}$

Hence, $x + 1 = \frac{p + q}{q}$ is a rational number and $x + 1 > x$

Thus, the assumption is false and there is no greatest negative rational number.

9 a) How do you prove "$\forall x \in D$, if $P(x)$ then $Q(x)$" by contraposition?

9 b) Prove: For all integers $a$, $b$, and $c$, if $a \nmid bc$ then $a \nmid b$.

This is a Counter example of the inverse:

$\forall a, b, c \in \mathbb{Z}$ if $a | bc$ then $a | b$

The inverse is not logically equivalent to the statement.

This is wrong!

Counter example:

Let $a = 2$, $b = 3$, and $c = 2$

$2 | 3 \cdot 2$ but $2 \nmid 3$
12 a) What are the steps for proving "For all integers \( n \) such that \( n \geq a \), \( P(n) \) is true" using mathematical induction?

12 b) Prove: For all integers \( n \geq 1 \),
\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.
\]

**Basis Step:** For \( n=1 \),
\[
\frac{1}{i=1} i = 1 = \frac{1(1+1)}{2}.
\]

**Inductive Step:** Let \( k \in \mathbb{N} \) and assume that \( \sum_{i=1}^{k} i = \frac{k(k+1)}{2} \).

\[
\sum_{i=1}^{k+1} i = (k+1) + \sum_{i=1}^{k} i = (k+1) + \frac{k(k+1)}{2} = \frac{2k+2}{2} + \frac{k(k+1)}{2} = \frac{k^2 + 3k + 2}{2} = \frac{(k+1)(k+2)}{2}
\]

**Hence,**
\[
\sum_{i=1}^{k} i = \frac{(k+1)(k+2)}{2}.
\]

\[\therefore \forall n \in \mathbb{N}, \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \text{ by induction}\]
Set Difference Law: For all sets $A$ and $B$,

$$A - B = A \cap B^c$$

\[
\begin{align*}
\text{Let } A, B \text{ be sets} \\
\text{Let } x \in A - B \\
\therefore x \in A \text{ and } x \notin B \\
\therefore x \in A \text{ and } x \in B^c \\
\therefore x \in A \cap B^c \\
\text{Hence, } A - B \subseteq A \cap B^c
\end{align*}
\]

This shows that $A - B \subseteq A \cap B^c$

We need to also show that $A \cap B^c \subseteq A - B$
21) Prove using the given set theory laws that for all sets $A$ and $B$,
\[ A \cup (B - A) = A \cup B \]

The Venn diagram helps us understand the problem, but it is not a formal proof.
22) Prove using the given set theory laws that for all sets $A$ and $B$, 

$$(B^c \cup (B^c - A))^c = B$$

Let $A, B$ be sets

$$
\left( B^c \cup (B^c - A) \right)^c = \left( B^c \cup (B^c \cap A^c) \right)^c \quad \text{By set different law} \\
= (B^c)^c \quad \text{By absorption law} \\
= B \quad \text{By double complement law}
$$

We need to justify each step by citing one of the set theory laws.