Trig Review

The Unit Circle has radius 1.

The angle $\Theta$ measured in radians is the angle so that the enclosed arc of the unit circle has length $\Theta$.

**Sohcahtoa**

- $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$
- $\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$
- $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin(\theta)}{\cos(\theta)}$

- $\csc(\theta) = \frac{1}{\sin(\theta)}$
- $\sec(\theta) = \frac{1}{\cos(\theta)}$
- $\cot(\theta) = \frac{1}{\tan(\theta)}$

**Pythagorean Theorem**

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$a^2 + b^2 = c^2$$
3.3 Derivatives of Trig Functions and Limits of Trig Functions

For unit circle
\[ \sin(\theta) = \frac{y}{1} = y \]

What is \( \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \)?

For very small \( \theta \), consider the picture.

For even smaller \( \theta \), consider

The arc with length \( \theta \), gets closer and closer to the opposite side with length \( \sin(\theta) \).

This gives:
\[ \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1 \]

Note: \( \sin(0) = 0 \)

Notation means the derivative of \( \sin(\theta) \) evaluated at \( \theta = 0 \)

\[ \left. \frac{d}{d\theta} \sin(\theta) \right|_{\theta=0} = \lim_{h \to 0} \frac{\sin(0+h) - \sin(0)}{h} = \lim_{h \to 0} \frac{\sin(h)}{h} = 1 \]

Note: \( \cos(0) = 1 \)

\[ \left. \frac{d}{d\theta} \sin(\theta) \right|_{\theta=0} = \cos(0) \]

Theorem: \( \frac{d}{d\theta} \sin(\theta) = \cos(\theta) \)

\( \theta \) must be measured in radians !!!!
\[ \cos(\theta) = \frac{x}{1} = x \]

What is \( \lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta} \)?

For small \( \theta \), consider the picture.

If \( \theta \) is small, then the arc with length \( \theta \), is much bigger than the segment with length \( 1 - \cos(\theta) \).

\[
\lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta} = 0
\]

\[
\frac{d}{d\theta} \cos(\theta) \bigg|_{\theta=0} = \lim_{h \to 0} \frac{\cos(\theta + h) - \cos(\theta)}{h}
\]

\[
= \lim_{h \to 0} \frac{\cos(h) - 1}{h}
\]

Note: \( \sin(\theta) = 0 \)

\[
= -\lim_{h \to 0} \frac{1 - \cos(h)}{h} = 0 = -\sin(\theta)
\]

**Theorem:** \( \frac{d}{d\theta} \cos(\theta) = -\sin(\theta) \)

\( \theta \) is measured in radians !!!!
\[
\frac{d}{dx} \tan(x) = \frac{d}{dx} \frac{\sin(x)}{\cos(x)}
\]
Recall: Quotient rule
\[
f = \frac{t}{b} \quad f' = \frac{t'b - b't}{b^2}
\]
\[
= \frac{\cos(x) \cdot \cos(x) - (-\sin(x)) \sin(x)}{\cos^2(x)}
\]
Pythagorean Theorem
\[
\cos^2(x) + \sin^2(x) = 1
\]
\[
= 1
\]
\[
\frac{1}{\cos^2(x)} = \sec^2(x)
\]

**Derivatives of trig functions**

1) \( \frac{d}{dx} \sin(x) = \cos(x) \)
2) \( \frac{d}{dx} \cos(x) = -\sin(x) \)
3) \( \frac{d}{dx} \tan(x) = \frac{1}{\cos^2(x)} = \sec^2(x) \)

\[
\frac{d}{dx} \csc(x) = \frac{d}{dx} \frac{1}{\sin(x)}
\]
Use Quotient rule to evaluate
\[
\frac{d}{dx} \sec(x) = \frac{d}{dx} \frac{1}{\cos(x)}
\]
\[
\frac{d}{dx} \cot(x) = \frac{d}{dx} \frac{1}{\tan(x)} = \frac{d}{dx} \frac{\cos(x)}{\sin(x)}
\]

**Example:** \( f(x) = \cos(x) \)

What is \( f^{(3\pi)}(x) \)?

\[
f(x) = \cos(x)
\]
\[
f'(x) = -\sin(x)
\]
\[
f''(x) = -\cos(x)
\]
\[
f'''(x) = -(-\sin(x)) = \sin(x)
\]
\[
f^{(4)}(x) = \cos(x)
\]
3\(\pi = 9.4 + 1\)

\[
f^{(3\pi)}(x) = \cos(x) \quad \text{because } 3\pi \text{ is a multiple of } 4
\]
\[
f^{(3\pi)}(x) = -\sin(x)
\]
Recall: \( \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1 \)

What is \( \lim_{\theta \to 0} \frac{\sin(5\theta)}{12\theta} \)?

\[ \lim_{\theta \to 0} \frac{\sin(5\theta)}{12\theta} = \frac{5}{12} \lim_{\theta \to 0} \frac{\sin(5\theta)}{5\theta} = \frac{5}{12} \cdot 1 = \frac{5}{12} \]

Recall: \( \lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta} = 0 \)

What is \( \lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta^2} \)?

\[ \lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta^2} = \lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta^2} \quad \frac{1 + \cos(\theta)}{1 + \cos(\theta)} \]

\[ = \lim_{\theta \to 0} \frac{1 - \cos^2(\theta)}{\theta^2 (1 + \cos(\theta))} \]

\[ = \lim_{\theta \to 0} \frac{\sin^2(\theta)}{\theta^2 (1 + \cos(\theta))} \]

\[ = \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \cdot \frac{\sin(\theta)}{\theta} \cdot \frac{1}{1 + \cos(\theta)} \]

\[ = 1 \cdot 1 \cdot \frac{1}{1 + 1} = \frac{1}{2} \]

\[ \lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta^2} = \frac{1}{2} \]