Math 266 Chapters 9 review problems

**Problem 1.** Let \( f : \mathbb{R} \to \mathbb{R} \) be the function defined by \( f(x) = x^2 + 1 \) \( \forall x \in \mathbb{R} \).

a. What is \( |f\{\{-1, 0, 1, 3\}\}| \)?

\[
|f\{\{-1, 0, 1, 3\}\}| = |\{-2, 1, 2, 10\}| = |\{1, 2, 10\}| = 3
\]

b. What is \( f([1, 3]) \)?

\[f([1, 3]) = [2, 10] \]

c. What is \( f((-1, 2)) \)?

\[f((-1, 2)) = [1, 5] \]

d. What is \( f^{-1}([5, 10]) \)?

\[f^{-1}([5, 10]) = [-3, 1] \cup [2, 3] \]

e. What is \( f^{-1}((0, 2)) \)?

\[f^{-1}((0, 2)) = (-1, 1) \]
Problem 2. Let \( f : X \to Y \) be a function and let \( A, B \subseteq X \). Prove that

\[
f(A \cup B) = f(A) \cup f(B)
\]

let \( y \in f(A \cup B) \)

\( \exists x \in A \cup B \) such that \( f(x) = y \)

\( x \in A \) or \( x \in B \)

\[
f(x) \in f(A) \cup f(B)
\]

Hence, \( f(A \cup B) \subseteq f(A) \cup f(B) \)

let \( y \in f(A) \cup f(B) \)

\( \exists x \in A \) or \( \exists x \in B \) such that \( f(x) = y \)

\( f(x) \in f(A) \) or \( f(x) \in f(B) \)

Hence, \( f(A) \cup f(B) \subseteq f(A \cup B) \)

Thus, \( f(A \cup B) = f(A) \cup f(B) \)

Problem 3. Let \( f : X \to Y \) be a function and let \( C, D \subseteq Y \). Prove that

\[
f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)
\]

let \( x \in f^{-1}(C \cap D) \)

\( f(x) \in C \cap D \)

\( f(x) \in C \) and \( f(x) \in D \)

\( x \in f^{-1}(C) \) and \( x \in f^{-1}(D) \)

\( x \in f^{-1}(C) \cap f^{-1}(D) \)

Hence, \( f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D) \)

Now, let \( x \in f^{-1}(C) \cap f^{-1}(D) \)

\( x \in f^{-1}(C) \) and \( x \in f^{-1}(D) \)

\( f(x) \in C \) and \( f(x) \in D \)

\( f(x) \in C \cap D \)

\( x \in f^{-1}(C \cap D) \)

Hence, \( f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D) \)

Thus, \( f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D) \)
Problem 4. Let $f : X \to Y$ be a function and let $A, B \subseteq X$. Prove that

$$f(A \cap B) \subseteq f(A) \cap f(B)$$

Let $y \in f(A \cap B)$

$\exists x \in A \cap B$ such that $f(x) = y$

$x \in A$ and $x \in B$

$f(x) \in f(A)$ and $f(x) \in f(B)$

$f(x) \in f(A) \cap f(B)$

$y \in f(A) \cap f(B)$

Thus, $f(A \cap B) \subseteq f(A) \cap f(B)$

Problem 5. Give an example of sets $X$ and $Y$, subsets $A, B \subseteq X$, and a function $f : X \to Y$ such that

$$f(A \cap B) \neq f(A) \cap f(B)$$

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = |x| \quad \forall x \in \mathbb{R}$$

Let $A = \{ -1, 3 \}$ and $B = \{ 1, 3 \}$

$f(A \cap B) = f(\emptyset) = \emptyset$

$f(A) \cap f(B) = f(\{ -1 \}) \cap f(\{ 1, 3 \}) = \{ 1 \} \cap \{ 1, 3 \} = \{ 1 \}$

Thus, $f(A \cap B) \neq f(A) \cap f(B)$
Problem 6. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is one-to-one but not onto.

Prove that $f$ is not onto.

\[
\text{Let } f(x) = e^x \quad \forall x \in \mathbb{R} \\
\text{Since } e^x > 0 \quad \forall x \in \mathbb{R} \\
\text{Thus, } 0 \notin f(\mathbb{R}) \\
\text{$f$ is not onto.}
\]

Problem 7. Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ which is onto but not one-to-one.

Prove that $g$ is not one-to-one.

\[
\text{Let } g(x) = \begin{cases} 
\ln(x) & x > 0 \\
0 & x \leq 0 
\end{cases} \\
g(-1) = 0 \text{ and } g(-2) = 0 \\
\text{Thus, $g$ is not one-to-one}
\]

Problem 8. Give an example of a function $f : \{a, b, c\} \to \{1, 2, 3, 4\}$ which is one-to-one.

Why does there not exist an one-to-one function from $\{1, 2, 3, 4\}$ to $\{a, b, c\}$?

\[
\text{There does not exist a one-to-one function from } \{1, 2, 3, 4\} \text{ to } \{a, b, c\} \\
because \left| \{1, 2, 3, 4\} \right| > \left| \{a, b, c\} \right|
\]

Problem 9. Give an example of a function $g : \{1, 2, 3, 4\} \to \{a, b, c\}$ which is onto. Why does there not exist an onto function from $\{a, b, c\}$ to $\{1, 2, 3, 4\}$?

\[
\text{There does not exist an onto function from } \{a, b, c\} \text{ to } \{1, 2, 3, 4\} \\
because \left| \{a, b, c\} \right| < \left| \{1, 2, 3, 4\} \right|
\]
Problem 10. Let $f: \mathbb{Z}_6 \to \mathbb{Z}_6$ be defined by

$$f([x]) = [3x] \quad \forall x \in \mathbb{Z}.$$

Prove that $f$ is a bijection and find $f^{-1}$.

- $f([0]) = [3 \cdot 0] = [0]$
- $f([1]) = [3 \cdot 1] = [3]$
- $f([2]) = [3 \cdot 2] = [1]$
- $f([3]) = [3 \cdot 3] = [4]$
- $f([4]) = [3 \cdot 4] = [2]$
- $f([5]) = [3 \cdot 5] = [0]$

Thus, $f$ is a bijection and $f^{-1}([x]) = [\frac{x}{3}] \quad \forall x \in \mathbb{Z}$.

Problem 11. Let $g: \mathbb{Z}_6 \to \mathbb{Z}_6$ be defined by

$$f([x]) = [3x] \quad \forall x \in \mathbb{Z}.$$

Prove that $g$ is not a bijection.

- $f([0]) = [3 \cdot 0] = [0]$
- $f([2]) = [3 \cdot 2] = [6] = [0]$

Thus, $f([0]) = f([2])$ but $[0] \neq [2]$. $f$ is not an injection, and hence is not a bijection.
Math266 Chapters 10 review problems

Problem 1. Give a bijection $f : [2, 4] \rightarrow [1, 5]$

\[ f(x) = 2x - 2 \]

\[ f(2) = 2 \]
\[ f(4) = 5 \]

\[ \Delta y = \frac{f(4) - f(2)}{4 - 2} = \frac{5 - 1}{2} = 2 \]

\[ f(x) = mx + b \]
\[ f(2) = 2m + b \]
\[ 5 = 4m + b \]
\[ b = -3 \]

Problem 2. Give a bijection $f : (-1, 0] \rightarrow [0, 3]$

\[ f(x) = 3x + 3 \]
\[ f(0) = 3 \]
\[ \Delta y = \frac{3 - 0}{-1 - 0} = -3 \]

\[ f(x) = mx + b \]
\[ f(0) = -3 \]
\[ b = 0 \]

\[ f(x) = -3x \]

Problem 3. Give a bijection $f : (0, \infty) \rightarrow \mathbb{R}$

\[ f(x) = \ln(x) \]

Problem 4. Prove that there is a bijection from $[0, 1]$ to $(0, \infty)$. (You do not need to give the bijection, just prove that it exists.)

Let $f : [0, 1] \rightarrow (0, \infty)$ be defined by

\[ f(x) = x + 1 \]

$f$ is an injection

Let $g : (0, \infty) \rightarrow [0, 1]$ be defined by

\[ g(x) = \frac{1}{1 + x} \]

$g$ is an injection

There exists an injection from $[0, 1]$ to $(0, \infty)$ and there exists an injection from $(0, \infty)$ to $[0, 1]$. Thus, there exists a bijection from $[0, 1]$ to $(0, \infty)$ by the Schroder-Bernstein Theorem.
**Problem 5.** Let \( X \) be a countable set. Prove that if \( Y \subseteq X \) then \( Y \) is countable. (Hint: \( Y \) is finite or \( Y \) is infinite)

If \( Y \) is finite, then \( Y \) is countable.

Assume that \( Y \) is infinite. Then \( X \) is infinite.

\( X \) is countably infinite is equivalent to \( X \) being able to be written as a sequence \( \{ x_i \}_{i=1}^{\infty} \).

\( Y \subseteq X \) implies that for each \( y \in Y \), \( y = x_k \) for some \( k \in \mathbb{N} \).

Thus, \( Y = \{ x_i \}_{i=1}^{\infty} \) and hence \( Y \) is countable.

**Problem 6.** Let \( X \) and \( Y \) be countably infinite sets. Prove that \( X \cup Y \) is countable.

**Note:** The text in the image is partially legible. It appears to be a sketch or a rough draft.

\[ X = \{ x_i \}_{i=1}^{\infty}, \quad \text{and} \quad Y = \{ y_i \}_{i=1}^{\infty}, \quad \text{because} \quad X \text{ and } Y \text{ are countable} \]

\[ X \cup Y = \{ x_1, y_1, x_2, y_2, x_3, y_3, \ldots \} \]

Hence, \( X \cup Y \) is countable.

**Problem 7.** Let \( X \) be an uncountable set and let \( Y \subset X \) be a countable subset. Prove that \( X - Y \) is uncountable.

Assume that \( X - Y \) is countable.

\[ X = (X - Y) \cup Y \]

Thus, \( X \) is the union of two countable sets and hence \( X \) is countable by Problem 6.

This is a contradiction.

Thus, \( X - Y \) is uncountable.
Problem 8. Prove that \( \mathbb{Z} \) is countable.

\[
\mathbb{Z} = \{ 0, 1, -1, 2, -2, 3, -3, \ldots \}
\]

Hence, \( \mathbb{Z} \) is countable.

Problem 9. Prove that \( \mathbb{N} \times \mathbb{Z} \) is countable.

We can order \( \mathbb{N} \times \mathbb{Z} \) as a sequence.
Thus, \( \mathbb{N} \times \mathbb{Z} \) is countable.

Problem 10. Prove that \( \mathbb{Q} \) is countable by either giving an injection from \( \mathbb{Q} \) to a countable set or giving a surjection from a countable set to \( \mathbb{Q} \).

Let \( f : \mathbb{N} \times \mathbb{Z} \to \mathbb{Q} \) be defined by

\[
f(m, n) = \frac{n}{m}
\]

\( f \) is a surjection because \( \mathbb{Q} = \{ \frac{a}{m} \mid a \in \mathbb{Z}, m \in \mathbb{N} \} \).

Thus, \( \mathbb{Q} \) is countable because \( \mathbb{N} \times \mathbb{Z} \) is countable.