Math266 Chapters 5,6,8 review problems

1) Let $X$ be a set and let $P(x)$ and $Q(x)$ be open sentences over $X$.
   a) How would you prove the following statement by contrapositive?
      \[ \forall x \in X, P(x) \implies Q(x) \]
   
   b) How would you prove the following statement by contradiction?
      \[ \forall x \in X, P(x) \implies Q(x) \]

   c) How would you prove the following statement by contradiction?
      \[ \exists x \in X, P(x) \lor Q(x) \]
2) Prove that $\sqrt{2}$ is irrational.

3) Prove that there is not a smallest rational number greater than 3.
4) Let $a \in \mathbb{Z}$, and let $P(n)$ be an open sentence over $\mathbb{Z}$.

a) What are the steps for proving the following statement by induction?

\[ \forall n \in \mathbb{Z} \text{ such that } n \geq a, P(n) \text{ is true.} \]

b) Explain why induction works. (You do not need to prove that it works, just give the idea)
5) Prove for all \( n \in \mathbb{N} \),
\[
\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}
\]

6) Prove for all \( n \in \mathbb{N} \),
\[
\sum_{i=1}^{n} i^3 = \frac{n^2(n + 1)^2}{4}
\]
7) Prove that $4 | (5^n - 1)$ for every $n \in \mathbb{N}$.

8) Define a sequence $(s_n)_{n=0}^{\infty}$ by $s_0 = 0$, $s_1 = 4$, and $s_n = 6s_{n-1} - 5s_{n-2}$ for all $n \in \mathbb{Z}$ with $n \geq 2$. Conjecture a formula for $s_n$ for all $n \in \mathbb{Z}$ with $n \geq 0$, and prove that the formula holds.
9) Prove whether or not the following relations on \( \mathbb{Z} \) satisfy the properties: reflexive, symmetric, or transitive.

a) \( xRy \iff |x - y| \leq 3 \)

b) \( xSy \iff |x - y| \geq 3 \)

c) \( xTy \iff x \equiv 4y(\text{mod } 3) \)
10) Let $X = \{a, b, c, d\}$

b) Give a relation on $X$ which is reflexive and symmetric but not transitive.

c) Give a relation on $X$ which is transitive, not symmetric, and not reflexive.

d) Give a relation on $X$ which is reflexive, not symmetric, and not transitive.

e) Give a relation on $X$ which is transitive and symmetric but not reflexive.
11) Write each of the following as \([r] \in \mathbb{Z}_7\), where \(0 \leq r < 7\)
   a) \([3] + [5] + [4]\)

   b) \([3][5] + [4]\)

   c) \([6]^{100}\) (Hint: it might be easier to raise a number equivalent to 6 to the 100 power)

   d) \([5]^{-1}\)

   e) \([6]^{-1}\)