1. Chapter 4 Review

1) Prove using the definition of even: For all integers \( n \), if \( n \) is even then \((-1)^n = 1\).

   \[ \begin{align*}
   \text{Let } n \in \mathbb{Z} \text{ be even.} \\
   \therefore \exists p \in \mathbb{Z} \text{ such that } n &= 2p \text{ by the definition of even.} \\
   \therefore (-1)^n &= (-1)^{2p} = (\{-1\}^2)^p = 1^p = 1 \\
   \therefore (-1)^n &= 1
   \end{align*} \]

2) Prove using the definition of odd: The product of any two odd integers is odd.

   \[ \begin{align*}
   \text{Let } n, m \in \mathbb{Z} \text{ be odd.} \\
   \therefore \exists p, q \in \mathbb{Z} \text{ such that } n &= 2p+1 \text{ and } m = 2q+1 \text{ by the definition of odd.} \\
   n \cdot m &= (2p+1)(2q+1) \\
   &= 4pq + 2p + 2q + 1 \\
   &= 2(2pq + p + q) + 1 \\
   \therefore n \cdot m \text{ is odd.}
   \end{align*} \]
3) Prove: For each integer \( n \) with \( 1 \leq n \leq 5 \), \( n^2 - n + 11 \) is prime.

   For \( n = 1 \), \( 1^2 - 1 + 11 = 11 \) is prime.
   For \( n = 2 \), \( 2^2 - 2 + 11 = 13 \) is prime.
   For \( n = 3 \), \( 3^2 - 3 + 11 = 17 \) is prime.
   For \( n = 4 \), \( 4^2 - 4 + 11 = 23 \) is prime.
   For \( n = 5 \), \( 5^2 - 5 + 11 = 31 \) is prime.

\[ \forall n \in \mathbb{Z} \text{ with } 1 \leq n \leq 5, \ n^2 - n + 11 \text{ is prime by exhaustion.} \]

4) Show that \( .123123123... \) is a rational number.

   Let \( x = .123123123... \)

   \[ 1000x = 123.123123... \]

   \[ 1000x - x = 123 \]

   \[ 999x = 123 \]

   \[ x = \frac{123}{999} \]

   \[ .123123... = \frac{123}{999} \text{ is a rational number.} \]

5) Prove using the definition of divides: For all integers \( a, b \), and \( c \), if \( a \) divides \( b \) and \( b \) divides \( c \), then \( a \) divides \( c \).

   Let \( a, b, c \in \mathbb{Z} \). Assume \( a \mid b \) and \( b \mid c \).

   \[ \exists p, q \in \mathbb{Z} \text{ such that } b = a \cdot p \text{ and } c = b \cdot q \text{ by the definition of divides.} \]

   \[ c = (a \cdot p) \cdot q \]

   \[ c = a \cdot (p \cdot q) \]

   \[ a \mid c \]
6) Evaluate 50 \text{ div } 4 \text{ and } 50 \text{ mod } 4.

\[ 50 = 12 \cdot 4 + 2 \]

\[ \therefore 50 \text{ div } 4 = 12 \quad \text{and} \quad 50 \text{ mod } 4 = 2 \]

7) Prove using the definition of \text{ mod}: For every integer \( p \), if \( p \text{ mod } 5 = 2 \) then \( 4p \text{ mod } 5 = 3 \).

Let \( p \in \mathbb{Z} \) such that \( p \text{ mod } 5 = 2 \)

\[ \because \exists n \in \mathbb{Z} \text{ such that } p = 5n + 2 \quad \text{by the definition of mod} \]

\[ \therefore 4p = 4 \cdot (5n + 2) \]

\[ \therefore 4p = 4 \cdot 5n + 8 \]

\[ \therefore 4p = 5(4n + 1) + 3 \]

\[ \therefore (4p) \text{ mod } 5 = 3 \]

8) Prove: For all integers \( n \), \( n^2 - n \) is even.

Let \( n \) be an integer

Case 1: Assume that \( n \) is even

\[ \therefore n^2 \text{ is even because the product of two even integers is even.} \]

\[ \therefore n^2 - n \text{ is even because the difference of two even integers is even.} \]

Case 2: Assume that \( n \) is odd

\[ \therefore n^2 \text{ is odd because the product of two odd integers is odd}. \]

\[ \therefore n^2 - n \text{ is even because the difference of two odd integers is even}. \]

\[ \therefore \forall \text{ integers } n, \ n^2 - n \text{ is even}. \]
9 a) How do you prove a statement by contradiction?

Assume that the statement is false. Then use a logical argument to reach a contradiction. Conclude that the statement is true.

9 b) Prove: There is no greatest negative rational number.

Assume that there is a greatest negative rational number.

\[ \exists p \in \mathbb{Q} \text{ such that } p < 0 \text{ and } \forall q \in \mathbb{Q} \quad q < p \]

\[ p < 0 \Rightarrow p < \frac{p}{2} < 0 \]

\[ \frac{p}{2} \in \mathbb{Q} \text{ because the quotient of two nonzero rational numbers is rational.} \]

\[ \therefore p \text{ is not the greatest negative rational number. by proof by contradiction.} \]

9 a) How do you prove \( \forall x \in D, \text{ if } P(x) \text{ then } Q(x) \) by contraposition?

Let \( x \) be an arbitrary, but fixed element of \( D \).

Assume that \( Q(x) \) is false. Prove that \( P(x) \) is false.

Conclude that \( \forall x \in D, \text{ if } P(x) \text{ then } Q(x) \).

9 b) Prove: For all integers \( a, b, \) and \( c, \) if \( a \nmid bc \) then \( a \nmid b \).

Let \( a, b, c \in \mathbb{Z} \) be arbitrary, but fixed.

Assume that \( a \mid b \).

\[ \exists p \in \mathbb{Z} \text{ such that } b = a \cdot p \]

\[ bc = a \cdot (c \cdot p) \]

\[ a \mid bc \]

\[ \therefore \forall x, b, c \in \mathbb{Z} \text{ if } a \nmid bc \text{ then } a \nmid b. \text{ by proof by contraposition.} \]
2. CHAPTER 5 REVIEW

a. Compute: \[ \sum_{i=2}^{5} i^2 = 2^2 + 3^2 + 4^2 + 5^2 \]

b. Compute: \[ \prod_{i=-5}^{-1} i = (-5)(-4)(-3)(-2)(-1) \]

\[ = -120 \]

c. Compute: \[ \sum_{i=1}^{100} (i^2 - (i-1)^2) = 1^2 - 0^2 + 2^2 - 1^2 + 3^2 - 2^2 + \ldots + 99^2 - 98^2 + 100^2 - 99^2 \]

\[ = 100 \cdot 1 \]

\[ = 100 \]

d. Compute: \[ \prod_{i=2}^{50} \frac{(i-1)}{(i+1)(i+2)} = \frac{2 \cdot 1}{3 \cdot 4} \cdot \frac{3 \cdot 2}{4 \cdot 5} \cdot \frac{4 \cdot 3}{5 \cdot 6} \cdot \frac{5 \cdot 4}{6 \cdot 7} \cdot \frac{48 \cdot 47}{50 \cdot 51} \cdot \frac{49 \cdot 48}{50 \cdot 51} \cdot \frac{50 \cdot 49}{51 \cdot 52} \]

\[ = \frac{2 \cdot 1 \cdot 3 \cdot 2}{50 \cdot 51 \cdot 51 \cdot 52} \]
2) a. Compute 5！
   = 5 × 4 × 3 × 2 × 1
   = 120

b. Compute \( \binom{100}{29} \)
   = \frac{100!}{29! \cdot (100-29)!}
   = \frac{100!}{29!}
   = \frac{100 - 29!}{29!} = 100

c. Compute \( \binom{10}{8} \)
   = \frac{10!}{8! \cdot (10-8)!}
   = \frac{10!}{8! \cdot 2!}
   = \frac{10 - 7 \cdot 8!}{8! \cdot 2} = \frac{10 - 9 \cdot 8!}{2} = 45

d. Expand \((x - 1)^5\):

   \[
   (\frac{5}{0}) \cdot x^5 \cdot (-1)^0 + (\frac{5}{1}) \cdot x^4 \cdot (-1)^1 + (\frac{5}{2}) \cdot x^3 \cdot (-1)^2 + (\frac{5}{3}) \cdot x^2 \cdot (-1)^3 + (\frac{5}{4}) \cdot x^1 \cdot (-1)^4 + (\frac{5}{5}) \cdot x^0 \cdot (-1)^5
   \]

   \[
   = \frac{5!}{5! \cdot 0!} \cdot x^5 \cdot \frac{5!}{4! \cdot 1!} \cdot x^4 \cdot \frac{5!}{3! \cdot 2!} \cdot x^3 \cdot \frac{5!}{2! \cdot 3!} \cdot x^2 \cdot \frac{5!}{1! \cdot 4!} \cdot x \cdot \frac{5!}{0! \cdot 5!}
   \]

   \[
   = \frac{5!}{5! \cdot 0!} \cdot x^5 \cdot \frac{5!}{4! \cdot 1!} \cdot x^4 + \frac{5!}{3! \cdot 2!} \cdot x^3 + \frac{5!}{2! \cdot 3!} \cdot x^2 + \frac{5!}{1! \cdot 4!} \cdot x + \frac{5!}{0! \cdot 5!}
   \]

   \[
   = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1
   \]
3 a) What are the steps for proving “For all integers \( n \) such that \( n \geq a \), \( P(n) \) is true” using mathematical induction?

**Basis case:** Prove that \( P(a) \) is true.

**Inductive step:** Let \( k \) be an integer such that \( k \geq a \)
Assume that \( P(k) \) is true.
Prove that \( P(k+1) \) is true

Conclude that \( P(n) \) is true for all integers \( n \) such that \( n \geq a \)

3 b) Prove: For all integers \( n \geq 1 \),

\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.
\]

**Basis case:**

\[
\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}
\]

When \( n = 1 \)

Let \( k \in \mathbb{Z} \) such that \( k \geq 1 \).

**Inductive hypothesis:** Assume that \( \sum_{i=1}^{k} i = \frac{k(k+1)}{2} \)

To show:

\[
\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}
\]

**Inductive step:**

\[
\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)
\]

\[
= \frac{k(k+1)}{2} + (k+1)
\]

by substituting the inductive hypothesis

\[
= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}
\]

\[
= \frac{(k+1)(k+2)}{2}
\]

\[
\therefore \forall n \in \mathbb{Z} \text{ with } n \geq 1, \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}
\]
4) Prove: For all integers \( n \geq 0 \),
\[
\sum_{i=1}^{n+1} i^2 = n^{2n+2} + 2
\]

**Basis case:**
\[
\sum_{i=1}^{2} i^2 = 2 = 02^{0+2} + 2
\]
\[
\therefore \sum_{i=1}^{n+1} i^2 = n^{2n+2} + 2 \quad \text{for } n = 0
\]

Let \( k \in \mathbb{Z} \) such that \( k \geq 0 \)

**Inductive Hypothesis:** Assume that \( \sum_{i=1}^{k} i^2 = k^{2k+2} + 2 \)

To show: \( \sum_{i=1}^{k+1} i^2 = (k+1)^{2(k+1)+2} + 2 \)

**Inductive Step:**
\[
\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2
\]
\[
= \sum_{i=1}^{k} i^2 + 2(k+1)^2 + 2 \quad \text{by substituting the induction hypothesis}
\]
\[
= (k^2 + 2) + 2(k+1)^2 + 2
\]
\[
= (2k+2) + 2(k+1)^2 + 2
\]
\[
= 2(k+1)^2 + 2
\]
\[
= (k+1)^{2k+2} + 2
\]
\[
= (k+1)^{2k+3} + 2
\]
\[
\therefore \forall \, n \in \mathbb{Z} \text{ such that } n \geq 0, \quad \sum_{i=1}^{n} i^2 = n^{2n+2} + 2 \quad \text{by induction}
6 a) What are the steps for proving “For all integers $n$ such that $n \geq a$, $P(n)$ is true” using strong mathematical induction? Fix an integer $b$ such that $b \geq a$.

**Basis Step.** Prove that $P(a), P(a+1), P(a+2), \ldots, P(b)$ are all true.

**Inductive Step.** Let $k \in \mathbb{Z}$ such that $k \geq b$. Assume that $P(a), P(a+1), \ldots, P(b), \ldots, P(k)$ are all true. Prove that $P(k+1)$ is true.

Conclude that $\forall n \in \mathbb{Z}$ such that $n > a$, $P(n)$ is true.

6 b) Prove: If $a_0 = 0$, $a_1 = 4$, and $a_k = 6a_{k-1} - 5a_{k-2}$ for $k \geq 2$, then $a_n = 5^n - 1$ for all integers $n \geq 0$.

**Basis Step:**

- $S_0 = 0^0 - 1 = 0 - 1 = -1$
- $S_1 = 4 - 5^1 - 1 = 4 - 5 - 1 = -2$

**Inductive Step:**

- Assume $S_n = 5^n - 1$ for all integers $n \geq 0$ such that $0 \leq n \leq k$.

To show: $S_{k+1} = 5^{k+1} - 1$

**Inductive Step:**

- Assume $k \geq 1$.
- $k+1 \geq 2$.

- $S_{k+1} = 6S_k - 5S_{k-1}$

- $= 6(5^k - 1) - 5(5^{k-1} - 1)$

- $= 6 \cdot 5^k - 6 - 5 \cdot 5^{k-1} + 5$

- $= 6 \cdot 5^k - 5 \cdot 5^{k-1} - 1$

- $= 6 \cdot 5^k - 5^{k+1} - 1$

- $= 5^{k+1} - 1$

- $\forall n \in \mathbb{Z}$ such that $n \geq 0$, $S_n = 5^{k+1} - 1$.