Our goal is to prove the Chinese Remainder Theorem.

**Theorem 1** (Chinese Remainder Theorem). Suppose $n_1, n_2, \ldots, n_L$ are positive integers that are pairwise relatively prime, that is, $(n_i, n_j) = 1$ for $i \neq j$, $1 \leq i, j \leq L$. Then the system of $L$ congruences

\[
\begin{align*}
x &\equiv a_1 \pmod{n_1} \\
x &\equiv a_2 \pmod{n_2} \\
& \quad \vdots \\
x &\equiv a_L \pmod{n_L}
\end{align*}
\]

has a unique solution modulo the product $n_1 n_2 \cdots n_L$.

We have proven already that the theorem is true when $L=2$. Before proving the general case, prove the following lemma.

**Lemma 1.** Suppose $n_1, n_2$ are positive integers such that $(n_1, n_2) = 1$. Suppose $x_0$ is a solution to the system of congruences

\[
\begin{align*}
x &\equiv a_1 \pmod{n_1} \\
x &\equiv a_2 \pmod{n_2}
\end{align*}
\]

Then $x_1$ is a solution as well, if and only if $x_1 \equiv x_0 \pmod{n_1 n_2}$.

Thus the solutions to the two given congruence equations are exactly the solutions to a single congruence equation. This idea of reducing the number of congruence equations may be useful in proving the Chinese Remainder Theorem. You may first want to try proving the Chinese Remainder Theorem for $L=3$ and $L=4$ to get a better feel for the problem before proving the general case.

The Chinese remainder theorem tells us when a solution exists, but it does not tell us how to find a solution. However, having a good understanding of the proof of the Chinese remainder theorem could be helpful in finding a solution.

**Question 1.** Suppose $n_1$ and $n_2$ are natural numbers with $(n_1, n_2) = 1$, and that $a_1$ and $a_2$ are integers. How can you find all integers $x$ that satisfy the following?

\[
\begin{align*}
x &\equiv a_1 \pmod{n_1} \\
x &\equiv a_2 \pmod{n_2}
\end{align*}
\]

Now that you can find the solutions to two congruences, you can use that solution to go further.

**Question 2.** Suppose $n_1, \ldots, n_L$ are natural numbers with $(n_i, n_j) = 1$ for all $i \neq j$, $1 \leq i, j \leq L$, and that $a_1, \ldots, a_L$ are integers. How can you find all integers $x$ that satisfy the following?

\[
\begin{align*}
x &\equiv a_1 \pmod{n_1} \\
x &\equiv a_2 \pmod{n_2} \\
& \quad \vdots \\
x &\equiv a_L \pmod{n_L}
\end{align*}
\]