It takes me 3 hrs to drive to College Station from Austin. If College Station is 180 miles away then what is my average velocity?

\[
\text{Average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{180 \text{ mi}}{3 \text{ hr}} = 60 \text{ mi/hr}
\]

How to determine instantaneous velocity?

Let \( f(t) \) denote the position of a moving object at time \( t \). \( f \) is called the position function.

The average velocity from time \( t = a \) to \( t = a + h \) is

\[
\text{displacement} = \frac{f(a + h) - f(a)}{h}
\]

The instantaneous velocity at time \( t = a \) is

\[
V(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

Ex: An apple is dropped from a height of 10m. The height of the apple at time \( t \) is given by the equation

\[f(t) = 10 - 4.9 t^2\]

What is the velocity of the apple after 1 second?

\[
V(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{10 - 4.9(1+h)^2 - (10 - 4.9)}{h}
\]

\[= \lim_{h \to 0} \frac{-9.8h - h^2}{h} = \lim_{h \to 0} -9.8 - h = -9.8 \text{ m/s}\]
3.2 The derivative as a function

If $f$ is a function then the derivative of $f$ is
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Other notation: If $y = f(x)$ then
$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{df}{dx} f(x) = Df(x) = D_x f(x)$$

Ex: $F(x) = x^2 - 2x + 1$  \hspace{1cm} f'(x) = 2x - 2

![Graphs]
How can a function fail to be differentiable at a point?

1. A corner or sharp point
   \[ y = f'(x) \]
   Ex: \( |x| \)

2. A discontinuity
   \[ y = f'(x) \]
   Ex: \( H(x) = \begin{cases} \sin(x) & \text{if } x < 0 \\ \sin^2(x) & \text{if } x \geq 0 \end{cases} \)

3. A vertical tangent line
   \[ y = f'(x) \]
   Ex: \( \sqrt{x} \)

Question: Does there exist a function \( f \) which is continuous at every point, but not differentiable at any point? *Yes!*

Define \( f(x) = \sum_{i=1}^{\infty} f_i(x) \)

\[ y = f_1(x) \]

\[ y = f_2(x) \]

\[ y = f_3(x) \]

\[ y = f_4(x) \]

\[ \vdots \]
Higher Derivatives

\( y = f(x) \) is a function

\[ \frac{dy}{dx} = f'(x) \] is the derivative

\[ \frac{d^2y}{dx^2} = f''(x) \] is the 2nd derivative (derivative of the derivative)

\[ \frac{d^3y}{dx^3} = f'''(x) \] is the 3rd derivative

\[ \frac{d^4y}{dx^4} = f^{(4)}(x) \] is the 4th derivative

Ex: \( S(t) \) is the position function

\( V(t) = S'(t) \) is the velocity

\( a(t) = V'(t) = S''(t) \) is the acceleration

\( j(t) = a'(t) \) is the jerk
3.3 Differentiation Formulas

1. Derivative of a constant function
   \[ \frac{d}{dx}(c) = 0 \]

[Diagram: \( y = c \), slope = 0]

2. \[ \frac{d}{dx}(x) = 1 \]

[Diagram: \( y = x \), slope = 1]

3. **Power rule** If \( n \) is any real number then
   \[ \frac{d}{dx}(x^n) = nx^{n-1} \]

4. **Constant multiple rule**
   \[ \frac{d}{dx}(cf(x)) = c \frac{d}{dx}f(x) \]

5. **Sum rule** If \( f \) and \( g \) are differentiable then
   \[ \frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \]

6. **Product rule** If \( f \) and \( g \) are differentiable then
   \[ \frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + f'(x)g(x) \]

7. **Quotient rule** If \( f \) and \( g \) are differentiable then
   \[ \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \]
Ex: \( f(x) = \frac{1}{x} = x^{-1} \)

\[ f'(x) = -x^{-2} \quad \text{by Power rule} \]

\[ f'(x) = \frac{x \cdot 0 - 1 \cdot 1}{x^2} \quad \text{by Quotient rule} \]

\[ = \frac{-1}{x^2} \]

\[ \frac{f'(x)}{g(x)} = f'(x) \cdot (g(x))^{-1} \]
Ex: \( f(x) = x^3 - 3x^2 + 6x - 7 \)

\[ g(x) = \frac{x^2 + 2}{x^2} \]

\[ h(x) = (x^2 + 2x + 1)(x + 3) \]

\[ F(t) = \sqrt{t} + t^3 - \sqrt[3]{t} \]

\[ g(t) = (\sqrt{t} + 1)^2 \]

\[ h(t) = \frac{t - 2}{t^2 - 1} \]

\[ V(x) = \frac{x + 1}{\sqrt{x} + 3} \]