Ex: \[ \lim_{x \to 2} \frac{x^2 - 5x + 6}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x - 3)}{x - 2} \]
\[= \lim_{x \to 2} (x - 3) = 2 - 3 = -1 \]

Ex: \[ \lim_{h \to 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} \]
\[= \lim_{h \to 0} \frac{6h + h^2}{h} \]
\[= \lim_{h \to 0} 6 + h = 6 \]

Ex: \[ \lim_{x \to 0} \frac{\sqrt{4+x} - 2}{x} = \lim_{x \to 0} \frac{x}{\sqrt{4+x} + 2} \]
\[= \lim_{x \to 0} \frac{4 + x - 4}{x(\sqrt{4+x} + 2)} \]
\[= \lim_{x \to 0} \frac{x}{x(\sqrt{4+x} + 2)} \]
\[= \lim_{x \to 0} \frac{1}{\sqrt{4} + 2} = \frac{1}{4} \]

Ex: \[ \lim_{t \to 0} \frac{\sqrt{t^2 + t} - 0}{t} = \lim_{t \to 0} \frac{t^2 + t - 0}{t(t^2 + t)} \]
\[= \lim_{t \to 0} \frac{t^2}{t(t^2 + 1)} \]
\[= \lim_{t \to 0} \frac{1}{t + 1} = 1 \]
Recall: A function $f$ is continuous at a point $a$ if
\[ \lim_{x \to a} f(x) = f(a) \]

Types of discontinuities

Removable discontinuity

\[ \lim_{x \to a} f(x) \text{ exists but } \lim_{x \to a} f(x) \neq f(a) \text{ or } f(a) \text{ does not exist} \]

Infinite discontinuity

$f$ has a vertical asymptote $x = a$

\[ \lim_{x \to a^-} f(x) = \pm \infty \text{ or } \lim_{x \to a^+} f(x) = \pm \infty \]

Jump discontinuity

\[ \lim_{x \to a^-} f(x) \text{ exists and } \lim_{x \to a^+} f(x) \text{ exists but } \lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) \]

NOTE: Not every discontinuous function is one of these types.
Def: A function \( f \) is continuous from the right at the value \( a \) if
\[
\lim_{x \to a^+} f(x) = f(a)
\]
and \( f \) is continuous from the left at \( a \) if
\[
\lim_{x \to a^-} f(x) = f(a)
\]

Ex: \( y = f(x) \)

\[
\lim_{x \to a^-} f(x) \neq f(a)
\]
so \( f \) is discontinuous from the left at \( a \)

\[
\lim_{x \to a^+} f(x) = f(a)
\]
so \( f \) is continuous from the right at \( a \)

Def: A function is continuous on an interval \( I \) (ex: \((-1, 1)\) or \([2, 5]\) or \((3, 7)\)) if \( f \) is continuous at every point in \( I \).

The Intermediate Value Theorem

If \( f \) is continuous on the closed interval \([a, b]\)
and \( N \) is some number between \( f(a) \) and \( f(b) \) where \( f(a) \neq f(b) \)
then there exists a number \( c \) in \((a, b)\) so that
\[
f(c) = N
\]
Ex! \[ f(x) = x^4 - 3x + 1 \]

\[ f(1) = 1^4 - 3 \cdot 1 + 1 = -1 \]
\[ f(2) = 2^4 - 3 \cdot 2 + 1 = 11 \]

By intermediate Value Theorem \[ f(x) = x^4 - 3x + 1 \] has a zero on the interval \((1, 2)\)
**Squeeze Theorem**

If \( f(x) \leq g(x) \leq h(x) \)

when \( x \) is near the value \( a \) (except possibly)

and \( \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \) at \( a \)

then \( \lim_{x \to a} g(x) = L \)

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**Example:** Find \( \lim_{x \to 0} x^2 \sin \left( \frac{1}{x} \right) \)

\(-1 \leq \sin \left( \frac{1}{x} \right) \leq 1\)

\(-x^2 \leq x^2 \sin \left( \frac{1}{x} \right) \leq x^2\)

\( \lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0 \)

Thus \( \lim_{x \to 0} x^2 \sin \left( \frac{1}{x} \right) = 0 \) by Squeeze thm.
3.1 Derivatives

Question: What is the tangent line?

What is the slope of the tangent line?

Idea: Approximate the tangent line by secant lines.

Def: If $f$ is a function then the derivative of $f$ at the value $a$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

or equivalently

$$f'(a) = \lim_{x \to a} \frac{f(a) - f(x)}{a-x}$$