6.1 Area between curves

Let $f$ and $g$ be continuous functions on $[a, b]$ such that $f(x) \geq g(x) \geq 0$ for all $x$ in $[a, b]$.

What is this area?

Subtract the area below $g$ from the area below $f$.

Ex: Find the area bounded by $y = x^{2} + 1$, $y = x$, $x = 0$, and $x = 1$.

\[
A = \int_{0}^{1} (x^{2} + 1 - x) \, dx
\]

\[
A = \left. \frac{1}{3} x^{3} + x - \frac{1}{2} x^{2} \right|_{0}^{1}
\]

\[
A = \frac{1}{3} \cdot 1^{3} + 1 - \frac{1}{2} \cdot 1^{2} - \frac{1}{3} \cdot 0^{3} + 0 - \frac{1}{2} \cdot 0^{2}
\]

\[
A = \frac{1}{3} + 1 - \frac{1}{2}
\]
Ex: Find the area between the parabolas: \( y = x^2 \) and \( y = 2x - x^2 \)

**Step 1:** To find where \( y = x^2 \) and \( y = 2x - x^2 \) intersect.

\[
x^2 = 2x - x^2 \quad \text{< set the } y \text{ values equal to each other}
\]

\[
0 = 2x - 2x^2 \quad \text{< solve for } x
\]

\[
0 = 2x(1-x)
\]

\[
x = 0, 1
\]

\[
2x - x^2 \geq x^2 \text{ whenever } 0 \leq x \leq 1
\]

\[
A = \int_0^1 (2x - x^2) - x^2 \, dx
\]

\[
A = \int_0^1 2x - 2x^2 \, dx
\]

\[
A = x^2 - \frac{2}{3} x^3 \bigg|_0^1
\]

\[
A = 1 - \frac{2}{3} - (0 - \frac{2}{3} 0)
\]

\[
A = 1 - \frac{2}{3}
\]

\[
A = \frac{1}{3}
\]
What if \( f(x) \neq g(x) \) for all \( x \in [a, b] \)?

Can consider the 3 areas separately!

\[
A_1 = \int_a^c f(x) - g(x) \, dx \\
A_2 = \int_c^d g(x) - f(x) \, dx \\
A_3 = \int_d^b f(x) - g(x) \, dx
\]

\[
A = A_1 + A_2 + A_3
\]

Can consider just (integral):

\[
k = \int_a^b |f(x) - g(x)| \, dx
\]

Ex: Find the area between

\[y = \sin(x), \quad y = \cos(x), \quad x = 0, \text{ and } x = \frac{\pi}{2}\]

\[
y = \sin(x) \\
y = \cos(x)
\]

If \( 0 \leq x \leq \frac{\pi}{4} \) then \( \sin(x) \geq \cos(x) \)

\[
A = \int_0^{\frac{\pi}{4}} \cos(x) - \sin(x) \, dx
\]

\[
= \sin(x) + \cos(x) \bigg|_0^{\frac{\pi}{4}} - \cos(x) - \sin(x) \bigg|_0^{\frac{\pi}{4}}
\]

\[
= \sqrt{2} - 1 - 1 + \sqrt{2} - 0
\]

\[
= 2\sqrt{2} - 2
\]
Ex: Find the area enclosed by the parabola \( y^2 = 2x + 6 \) and the line \( y = x - 1 \)

Here \( x \) is a function of \( y \)!

\[
\begin{align*}
y^2 &= 2x + 6 \\
x &= \frac{1}{2} y^2 - 3
\end{align*}
\]

\[
y = x - 1 \\
x &= y + 1
\]

**Step 1: Find the intersection points**

\[
\begin{align*}
y + 1 &= \frac{1}{2} y^2 - 3 \\
0 &= \frac{1}{2} y^2 - y - 4 \\
0 &= \frac{1}{2} y^2 - y - 4 \\
0 &= (y - 4)(y + 2)
\end{align*}
\]

\( y = -2, 4 \)

\[
\begin{align*}
x &= \frac{1}{2} y^2 - 3 \\
y = \frac{4}{2} = 2
\end{align*}
\]

\[
\int_{-2}^{4} \left( y + 1 - \left( \frac{1}{2} y^2 - 3 \right) \right) dy
\]

\[
\int_{-2}^{4} \left( \frac{1}{2} y^2 + y + 4 \right) dy
\]

\[
\int_{-2}^{4} \left( -\frac{1}{6} y^3 + \frac{1}{2} y^2 + 4y \right) dy
\]

\[
A = 18
\]
6.2 Volume

A cylinder is a solid whose cross section is always the same.

Volume of Cylinder

\[ V = A \cdot l \]

**Example:**

Area of cross section: \( h \cdot w \)

\[ V = A \cdot l \]

\[ V = h \cdot w \cdot l \]
What is the volume?

Idea: Approximate the object with cylinders similar to how area can be approximated with rectangles.

Idea: If the length of each approximating cylinder is small then the volume of the solid will be very close to the sum of the volumes of the cylinders.

\[ V_{cy} = A(x_1) \Delta x + A(x_2) \Delta x + A(x_3) \Delta x + \cdots + A(x_n) \Delta x \]

If \( x \) is in \([a, b]\) let \( A(x) \) be the area of the cross section of the solid at \( x \) then

\[ V = \int_a^b A(x) \, dx = \lim_{n \to \infty} A(x_1) \Delta x + A(x_2) \Delta x + \cdots + A(x_n) \Delta x \]