Recall: If \( f \geq 0 \) is a positive continuous function on an interval \([a, b]\) then \( \int_a^b f(x) \, dx \) is the area under the curve \( y = f(x) \) and above the interval \([a, b]\).

Let \( g(x) = \int_a^x f(t) \, dt \).
The Fundamental Theorem of Calculus (Part 1)

Let $f$ be a continuous function on an interval $[a,b]$

Define $g(x) = \int_a^x f(t) \, dt$

Then $g$ is continuous on $[a,b]$
$g$ is differentiable on $(a,b)$
$g'(x) = f(x)$ for all $x$ in $(a,b)$

Ex: $g(x) = \int_0^x \sqrt{1+t^2} \, dt$

$g'(x) = \sqrt{1+x^2}$

Ex: $h(x) = \int_0^x x^3 \sqrt{1+t^2} \, dt$

Let $g(x) = \int_0^x \sqrt{1+t^2} \, dt$

Then $h(x) = g(x^3)$

$h'(x) = g'(x^3) \cdot 3x^2$ by the chain rule

$h''(x) = \frac{\sqrt{1+(x^3)^2}}{3x^2} - 3x^2$ by FTC

Ex: $h(x) = \int_x^5 \sqrt{1+t^2} \, dt$

$\frac{d}{dx} h(x) = \int_5^x \sqrt{1+t^2} \, dt = -\int_x^5 \sqrt{1+t^2} \, dt$

$h'(x) = -\sqrt{1+x^2}$ by FTC

Ex: $h(x) = \int_0^{\cos(x)} \sqrt{1+t^2} \, dt$

$h(x) = \int_0^{\cos(x)} \sqrt{1+t^2} \, dt + \int_{\cos(x)}^0 \sqrt{1+t^2} \, dt$

$h(x) = \int_0^{\cos(x)} \sqrt{1+t^2} \, dt - \int_{\cos(x)}^0 \sqrt{1+t^2} \, dt$

Let $g(x) = \int_0^{\cos(x)} \sqrt{1+t^2} \, dt$

Then $h(x) = g(\cos(x)) - g(\sin(x))$

$h'(x) = g'(\cos(x)) \cdot (-\sin(x)) - g'(\sin(x)) \cdot \cos(x)$ by chain rule

$h'(x) = \sqrt{1+\cos^2(x)} \cdot (-\sin(x)) - \sqrt{1+\sin^2(x)} \cdot \cos(x)$ by FTC
The Fundamental Theorem of Calculus (Part 2)

If \( f \) is continuous on \([a, b]\) then

\[
\int_a^b f(x) \, dx = F(b) - F(a) = F(x) \bigg|_a^b
\]

where \( F \) is any antiderivative of \( f \).

Ex: \( \int_{-2}^1 x^2 \, dx \)

\( \frac{1}{3}x^3 \) is an antiderivative of \( x^2 \)

\[
\int_{-2}^1 x^2 \, dx = \frac{1}{3}x^3 \bigg|_{-2}^1
= \frac{1}{3}(1)^3 - \frac{1}{3}(-2)^3
= \frac{1}{3} + \frac{8}{3}
= \frac{9}{3}
= 3
\]

Ex: \( \int_{-2}^1 \frac{1}{x^2} \, dx \)  \( \text{note that} \ \frac{1}{x^2} \ \text{is discontinuous at} \ x=0! \)

\( F \) must be continuous to use FTC

Ex: An object is moving with a velocity of \( v(t) \) at time \( t \).
How far does the object travel from time \( t=a \) to \( t=b \)?

Let \( s(t) \) denote the position at time \( t \).

The distance traveled from time \( t=a \) to \( t=b \) is the change in position: \( s(b) - s(a) \)

Position is an antiderivative of velocity so we apply FTC

\[
\int_{a}^{b} v(t) \, dt = s(b) - s(a) \text{ is how far the object traveled from time } t=a \text{ to } t=b
\]
Ex: An object is moving at $t+1$ m/s at time $t$. How far does the object travel between time $t=1$ and $t=3$?

$$S_1^{t+1} \, dt = \frac{1}{2} t^2 + t |_1^3$$

$$= \frac{1}{2} (3)^2 + 3 - \left( \frac{1}{2} (1)^2 + 1 \right)$$

$$= \frac{9}{2} + 3 - \frac{1}{2} \cdot 1$$

$$= \frac{9}{2} + 6$$

$$= 4 + 2 = 6$$

Ex: $\int_1^9 \frac{x-1}{\sqrt{x}} \, dx = \int_1^9 \frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}} \, dx$

$$= \int_1^9 x^{1/2} - x^{-1/2} \, dx$$

$$= \left. \frac{2}{3} x^{3/2} - 2 x^{1/2} \right|_1^9$$

$$= \frac{2}{3} (9)^{3/2} - 2 (9)^{1/2} - \left( \frac{2}{3} (1)^{3/2} - 2 (1)^{1/2} \right)$$

$$= \frac{2}{3} \cdot 27 - 2 \cdot 3 - \frac{2}{3} + 2$$

$$= 18 - 6 - \frac{2}{3} + 2$$

$$= 14 - \frac{2}{3}$$

Ex: $\int_2^4 \frac{x^2 - 1}{x + 1} \, dx = \int_2^4 \frac{(x+1)(x-1)}{x+1} \, dx$

$$= \int_2^4 (x-1) \, dx$$

$$= \left. \frac{1}{2} x^2 - x \right|_2^4$$

$$= \frac{1}{2} (4)^2 - 4 - \left( \frac{1}{2} (2)^2 - 2 \right)$$

$$= 8 - 4 - 2 + 2$$

$$= 4$$

Ex: $\int_0^{\pi/2} \sqrt{1 - \sin^2(x)} \, dx = \int_0^{\pi/2} \cos(x) \, dx$

$$= \left. \sin(x) \right|_0^{\pi/2}$$

$$= \sin(\pi/2) - \sin(0)$$

$$= 1 - 0$$

$$= 1$$
Find $h'(x)$ for

1. $h(x) = \int_1^x \cos^2(t) \, dt$ \hspace{1cm} h(x) = \cos^2(x)$

2. $h(x) = \int_{-1}^x \cos^2(t) \, dt$ \hspace{1cm} h'(x) = -\cos^2(x)$

3. $h(x) = \int_x^{x^3} \cos^2(t) \, dt$ \hspace{1cm} h'(x) = \cos^2(x^3) \cdot 3x^2 - \cos^2(x^3) \cdot 2x$

Evaluate

4. $\int_{-1}^1 x^2 - 1 \, dx = \frac{5\sqrt{3}}{3}$

5. $\int_0^{2\pi} \cos(\theta) \, d\theta = 0$

6. $\int_1^2 \frac{x^2 + 1}{\sqrt{x}} \, dx = \frac{2}{3} \cdot 2^{3/2} + 2 \cdot 2^{3/2} - \frac{2}{3} - 2$

7. $\int_1^2 \frac{x^2 + 2x + 1}{x + 1} \, dx = 3 - \frac{1}{2}$

8. $\int_0^1 (y + 1)(y + 2) \, dy = \frac{1}{3} + \frac{3}{2} + 2$

9. $\int_0^{\pi/2} \sin^3(\theta) + \sin(\theta) \cos^2(\theta) \, d\theta = 1$