Math/Stat 3850 – Final Exam Practice Questions

There are 15 questions, worth a total of 150 points. You may use R, your calculator, and any written or internet resources on this test, although you are not allowed to ask someone else for help.

1. (10) The Intel i5 and i7 computer processors are actually the same chip. Intel makes a wafer of 190 chips, then tests them for speed. About 60% of these chips test well and are sold as i7 chips, the rest are either discarded or sold as i5 chips.

   (a) What is the expected number of i7 chips on a wafer?
   (b) What is the probability that more than 120 chips on a wafer can be sold as i7’s?

Solution: (a) 114 (b) 0.168

2. (10) In some role playing games, character abilities are determined by rolling four six-sided dice, removing the smallest value, and adding the other three. For example, a roll of 2, 3, 5, 5 gives an ability 13.

   (a) What is the probability of rolling an ability of 13 or less?

   Write your code here, and give an approximate answer to within 0.01

   (b) A character has six different abilities, and if none of them are above 13 the character is discarded. What is the probability the character will be discarded?

Solution: (a) .645 (b) depending on the value in part a, around 0.068-0.082

3. (10) Let $X$ be a random variable with the PDF $f(x) = \begin{cases} c(1-x^2) & \text{for } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

   (a) Find $c$
   (b) Sketch the cumulative distribution function (CDF) for $X$.

Solution: $c = 3/4$. The CDF is

\[ F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{1}{2} + \frac{3}{4}x - \frac{1}{4}x^3 & \text{for } -1 \leq x \leq 1 \\ 1 & \text{for } 1 < x \]
4. With $X$ as in question 3:
   (a) What is $P(X > 0)$?
   (b) What is $P(X \leq \frac{1}{2})$?
   (c) What is $E(X)$?

Solution: (a) $\frac{1}{2}$, (b) $\frac{27}{32}$, (c) 0.

5. In the game of Scrabble, players draw tiles randomly from a bag. Tiles have different values, given by the probability distribution:

<table>
<thead>
<tr>
<th>Tile value</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.02</td>
<td>0.68</td>
<td>0.07</td>
<td>0.08</td>
<td>0.10</td>
<td>0.01</td>
<td>0</td>
<td>0</td>
<td>0.02</td>
<td>0</td>
<td>0.02</td>
</tr>
</tbody>
</table>

Compute the expected value of a Scrabble tile.

Solution: 1.87

6. Exact or approximate answers are acceptable.
   (a) Suppose $X$ has an exponential distribution with $\lambda = 0.1$. What is the expected value of $X$? What is the variance of $X$?
   (b) If $X_1, \ldots, X_{25}$ are i.i.d with $X_i \sim \text{Exp}(0.1)$, what is the expected value of $\bar{X} = \frac{1}{25} \sum_i X_i$? What is the variance of $\bar{X}$?

Solution: a. $\mu(X) = 10$. $\text{var}(X) = 100$.
b. $\mu(\bar{X}) = 10$. $\text{var}(\bar{X}) = \text{var}(X)/25 = 4$. 
7. Suppose your population has the exponential distribution with \( \lambda = 0.1 \), and you take a sample of size 25. You wish to test \( H_0: \mu = 10 \) versus \( H_0: \mu \neq 10 \) at the \( \alpha = .05 \) level.

(a) Approximate the type I error if you use a t-test.
(b) Approximate the type I error if you use a Wilcoxon test.
(c) What should the type I error be for these tests? Which test is performing closer to its designed behavior?

Solution: a. The t-test rejects \( H_0 \) approximately 7.5% of the time.
   b. The Wilcoxon test rejects \( H_0 \) approximately 14% of the time.
   c. Both tests should reject \( H_0 \) about 5% of the time. The t-test is performing closer to its designed behavior (though neither does particularly well with this strongly skewed population).

8. The dataset GAGurine from library(MASS) gives the concentration of the chemical GAG in the urine of children at different ages. The aim of the study was to produce a chart to help a pediatrician to assess if a child’s GAG concentration is typical.

(a) Is GAG concentration positively or negatively correlated with age?
(b) Would it be appropriate to describe the relation between age and GAG with a linear model? Why or why not?
(c) Fit a linear model to explain the logarithm of GAG from age. What is the equation of the line?
(d) Does the linear model accurately describe the relation between age and log(GAG)?

Solution: a. Negative - it goes down when age goes up. b. No, the relationship is clearly curved. c. \( \text{log(GAG)} = 2.9661 - 0.1139 \text{Age} \). d. No, the line is still not a good fit. The residuals show a definite curving pattern.

9. The genotype data set from library(MASS) gives the weight gain of baby mice raised by foster mothers. The foster mother genotype is given by the Litter variable.

Test the hypothesis that all litters have the same mean weight gain. Report the \( p \)-value with your conclusion.

Solution: anova(lm(Wt ~ Litter, data=genotype)) gives \( p = 0.8375 \). This data shows no effect of foster mother genotype on weight gain.

10. The immer data set from library(MASS) gives barley yield from various fields in 1931 (Y1) and 1932 (Y2). You want to compare the yield between the two years.
(a) What are the mean yields for each year?
(b) State $H_0$ and define any parameters you need to use.
(c) Carry out an appropriate hypothesis test, and summarize the results.

**Solution:**

a. Year 1 was 109.05, Year 2 was 93.13.

b. $H_0 : \mu_1 = \mu_2$, where $\mu_i$ is the true mean yield for year $i$.

c. `t.test(immer$Y1, immer$Y2, paired=TRUE)` gives a $p$-value of 0.002413. There was a significant difference in barley yield between those two years.

11. Suppose a population is normally distributed with mean 105 and standard deviation 15. You sample 40 data points and wish to test $H_0 : \mu = 100$ versus $H_0 : \mu \neq 100$ at the $\alpha = 0.05$ level.

(a) Approximate the type II error if you use a $t$-test.

**Solution:** Type II error happens about 46% of the time.

```r
mean(replicate(10000, t.test(rnorm(40, 105, 15), mu=100)$p.value >= 0.05))
```

(b) Approximate the type II error if you use a Wilcoxon test.

**Solution:** Type II error happens about 48% of the time.

```r
mean(replicate(10000, wilcox.test(rnorm(40, 105, 15), mu=100)$p.value >= 0.05))
```

(c) Which test is more powerful in this situation?

**Solution:** The $t$-test is more powerful.

12. Suppose $X_1, \ldots, X_{10}$ are ten independent random variables, each with the $t$ distribution with 5 df. Let $X$ be the maximum of these:

$$X = \max(X_1, X_2, \ldots, X_9, X_{10})$$

Estimate the mean and standard deviation of $X$ to within 0.1. Discuss the shape of the distribution of $X$.

**Solution:** $\mu_X \approx 2.0, \sigma_X \approx 1.13$. The distribution is bump shaped with a right skew, but may contain some extremely large outliers that make it hard to visualize.

```r
You can simulate $X$ with `x <- replicate(10000, max(rt(10, 5)))`
```
The “Bechdel Test” asks whether a movie features at least two women or girls who talk to each other about something other than a man or boy. The data set `bechdel.csv` on our course website has data on 1776 films from 1970-2013.

For questions 13-15, the relevant variables are:

- **test**: either “PASS” or “FAIL” depending on if the film passed the Bechdel test
- **reason**: the reason why the film failed the test, or “ok” if it passed.
- **budget**: the budget (in US$2013) for the film

13. (10) State and carry out a $t$-test that the mean budget is the same for movies that pass and movies that fail the Bechdel test. Report the $p$-value in your conclusions.

**Solution:**

$H_0 : \mu_P = \mu_F$, where $\mu_P$ and $\mu_F$ are the mean budgets of films that pass/fail the Bechdel test. $H_a : \mu_P \neq \mu_F$. Apply the `t.test` function as:

```
t.test(budget ~ test, data=bechdel)
```

The approximately $17$ million difference in budget is significant, with $P = 7.6 \times 10^{-11}$. Movies that fail the Bechdel test have higher budgets.

14. (10) Use ANOVA to check for a relationship between **budget** and the reason a film failed the test. Make sure you omit the films with reason ‘ok’ from your analysis – they passed.

**Solution:**

```
anova(lm(budget ~ reason, data=filter(bechdel, reason != 'ok')))
```

There is not a significant difference in budget among the various reasons for failing the Bechdel test ($P = 0.096$).

15. (10) Describe the shape of the distribution of the **budget** variable. This may cast doubt on the conclusions of problems 13 and 14. What could you do about it?

**Solution:** The budget distribution is very skew. With 1776 data points, our analysis is probably reasonable, but it would be wise to redo the tests after applying a log transformation to budget.