1. Show $10^{-n^2}$ converges linearly to 0.

Solution:

$$\lim_{n \to \infty} \frac{10^{-(n+1)^2}}{10^{-n^2}} = \lim_{n \to \infty} 10^{n^2-(n+1)^2} = \lim_{n \to \infty} 10^{-2n-1} = 0.$$  
Since this limit is less than 1, the sequence converges linearly to 0.

2. Give an example of a non-constant polynomial $g(x)$ so that iterating $x_0 = 1.7, x_1 = g(x_0), x_2 = g(x_1), \ldots$ converges to $\sqrt{5}$.

Solution: We’re looking for a root of $f(x) = x^3 - 5$. With Newton’s method, $g(x) = x - \frac{x^3 - 5}{3x^2}$. Another approach is to put $g(x) = x + cf(x)$ for some $c$ which makes $g'(\sqrt{5}) < 1$. For example $g(x) = x - 0.1(x^3 - 5)$ works.

3. The function $f(x) = x^2 - x - 1$ has roots $\varphi = \frac{1+\sqrt{5}}{2}$ and $\overline{\varphi} = \frac{1-\sqrt{5}}{2}$. The larger root $\varphi \approx 1.618$ is known as the golden mean. Use Newton’s method with $x_0 = 2$ to compute $x_1$ and $x_2$. However, do not use decimal approximations – carry out your computations exactly so that you get fractions which are good approximations to $\varphi$.

Solution:

$$g(x) = x - \frac{f(x)}{f'(x)} = \frac{x^2 + 1}{2x - 1}.$$  
$$g(2) = 5/3 \quad g(5/3) = 34/21$$  
Note that these will always be Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, . . . .
4. Let \( f(x) = x^3 - x \), as shown below. Apply Newton’s method to solve \( f(x) = 0 \).

(a) Begin with initial guess \( x_0 = 0.3 \). Which root does Newton’s method converge to?
(b) Begin with initial guess \( x_0 = 0.8 \). Which root does Newton’s method converge to?
(c) Begin with initial guess \( x_0 = 0.5 \). Which root does Newton’s method converge to?
(d) Begin with initial guess \( x_0 = 0.455 \). Which root does Newton’s method converge to?

**Solution:** a. 0; b. 1; c. -1; d. 1

5. Behold a function \( f(x) \) graphed below, along with the line \( y = x \). Draw an accurate cobweb plot starting at \( x_0 = 0.2 \), to see what will happen if you iterate this function.