**Problems:**

1. The number 99 was multiplied by an integer \( k \) to obtain an integer of seven decimal digits, but two of the digits got blotted out on the paper. The product was 62ab427, but the digits \( a \) and \( b \) are illegible. Determine all possible values of \( a \) and \( b \).


2. Find the sum of the series

   \[
   1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots,
   \]

   where the terms are the reciprocals of the positive integers whose only prime factors are twos and threes.


3. Notice that 73 can be written as a sum of two consecutive positive integers, 73 = 36 + 37. Prove that no longer sum of consecutive positive integers equals 73.

4. Suppose that the function \( f \) satisfies \( f'(x) = 1 + f(x) \) for all \( x \). If \( f(2) = 3 \) find:

   (a) \( f^{(10)}(2) \) where \( f^{(10)} \) denotes the 10th derivative of \( f \);
   (b) \( f(3) \).


5. For real \( a > 0 \) define the sequence \( \{x_n\} \) by

   \[
   x_{n+1} = a(x_n^2 + 4), \quad x_0 = 0.
   \]

   Determine necessary and sufficient conditions on \( a \) for \( \lim_{n \to \infty} x_n \) to exist and be finite.

1. Since the product ends in 27, the product is 73 less than a multiple of 100.
   This means \( k \equiv 73 \mod 100 \).

   Also,
   \[
   \frac{62626}{99} = \frac{620000}{994} \quad \text{so} \quad k \geq 62626 \quad \text{and} \quad \frac{62514}{99} = \frac{630000}{994}.
   \]

   Therefore, \( 62626 \leq k \leq 63514 \).

   The first possibility is 62673
   \[
   \begin{align*}
   62673 & \div 99 \quad = 664 \text{ remainder } 97, \\
   569057 & \div 6204627 \quad = 91 \text{ remainder } 97.
   \end{align*}
   \]

   Other possibilities are 100, 200, etc., to which increases the product by 9900 each time:
   \[
   \begin{align*}
   6204627 & \times 6254127, \\
   6214527 & \times 6264027, \\
   \underline{6224427} & \times 6273927, \\
   6234327 & \times 6283827, \\
   6244227 & \times 6293727.
   \end{align*}
   \]

   Only 6224427 fits, so \( ab \) can only be 24.

2. Let \( S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \ldots \).

   Then
   \[
   \frac{S}{n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{12} + \frac{1}{16} + \frac{1}{18} + \frac{1}{24} + \ldots,
   \]

   which includes any terms in \( S \) except reciprocals of \( 3^k, \ k \geq 0 \).

   Hence,
   \[
   \frac{S}{n} = S - \left[ \left( \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots \right) \right] = S - \frac{\frac{1}{2}}{1 - \frac{1}{3}} = \frac{3}{2}
   \]

   which implies
   \[
   \frac{S}{n} = \frac{3}{2} \quad \text{and} \quad S = 3.
   \]
If 73 is a sum of $k$ consecutive positive integers starting with $x$ then
\[ x + (x+1) + \ldots + (x+k-1) = kx + \frac{(k-1)k}{2} = 73 \]
If this equation holds then $73 - \frac{(k-1)k}{2} \equiv 0 \pmod{k}$, which we can check without knowing $x$.

Notice if $k$ is odd then $kx + \frac{(k-1)k}{2} \equiv 0 \pmod{k}$, while $73 \not\equiv 0 \pmod{k}$ for $k \geq 3$ because 73 is prime. So odd $k \geq 3$ is impossible.

Notice if $k=1$ and $k=2$ then $kx + \frac{k-1}{2}k = 78 > 73$. So only $k \leq 10$ need be considered.

Finally, we compute $73 - \frac{(k-1)k}{2} \equiv 0 \pmod{k}$ for $k = 4, 6, 8, 10$:

<table>
<thead>
<tr>
<th>$k$</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{mod}(73 - \frac{(k-1)k}{2}, k)$</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>8</td>
</tr>
</tbody>
</table>

Since none are zero, 73 cannot be written as a sum of $k$ consecutive positive integers for any $k \geq 2$.

4. Let $y = f(x)$ and rewrite the DE as $\frac{dy}{dx} - y = 1$.

The characteristic equation is $r - 1 = 0$ so $e^x$ is a homogeneous solution. Using the method of undetermined coefficients, we assume $y(x) = b$ is a particular solution.

\[ (b)' = b \rightarrow b = -1 \]

Thus the general solution is $y(x) = Ae^x - 1$.

Since $y(2) = 3$, we find that $3 = Ae^2 - 1$ so $A = 4e^{-2}$.

(a) Since $f(x) = 4e^2e^x - 1 \Rightarrow f(0)(x) = 4e^{-2}e^x \Rightarrow f(0)(2) = 4$.

(b) $f(3) = 4e^2e^3 - 1 = 4e^{-1}$.

5. If a limit exists then $x = 2x^2 + 4a \Rightarrow 2x^2 - x + 4a = 0$ has a real solution.

Thus a necessary condition is $1 - 16a^2 > 0$ or $a^2 < \frac{1}{16}$ or $0 < a < \frac{1}{4}$.

If $0 < a < \frac{1}{4}$ then the solutions are $x = \frac{1}{2a} \pm \sqrt{\frac{1 - 16a^2}{2a}}$ so $0 < x < \frac{1}{a}$.

1. $x_{n+1} - x_n = a(x_n^2 - x_{n-1}^2) \Rightarrow x_n - x_0 = 4a \geq 0$ so induction implies $x_{n+1} \geq x_n$.

2. $x_1 = 4a < 2 \Rightarrow x_{n+1} = a(x_n^2 + 4) \leq 2(4 + 4) = 16 \Rightarrow x_n \leq 2$.

Since $x_n$ is bounded, monotonic it must converge.