Mathematical Induction

The natural numbers are the counting numbers: 1, 2, 3, 4, .... Mathematical induction is a technique for proving a statement - a theorem, or a formula - that is asserted about every natural number. For example,

\[ 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}. \]

This asserts that the sum of consecutive numbers from 1 to \( n \) is given by the formula on the right. We want to prove that this will be true for \( n = 1, n = 2, n = 3, \) and so on. Now we can test the formula for any given number, say \( n = 3: \)

\[ 1 + 2 + 3 = \frac{3 \times 4}{2} = 6, \]

which is true. It is also true for \( n = 4: \)

\[ 1 + 2 + 3 + 4 = \frac{4 \times 5}{2} = 10. \]

But how are we to prove this rule for every value of \( n? \) The method of proof is the following:

**Principle of Mathematical Induction.**

*Suppose*

1) (The base case) The statement is true for \( n = 1; \)

2) If the statement is true for \( n \), then it is also true for \( n + 1; \)

*Then the statement is true for every natural number \( n. \)*

When the statement is true for \( n = 1, \) then according to 2), it will also be true for \( n = 2. \) But that implies it will be true for \( n = 3; \) which implies it will be true for \( n = 4. \) And so on. It will be true for every natural number. To prove a statement by induction, then, we must prove parts 1) and 2) above.

The hypothesis of part 2) - "The statement is true for \( n\)" - is called the inductive assumption, or the inductive hypothesis. It is what we assume when we prove a theorem by induction.

**Example 1.** Show that

\[ 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}. \] (1)

*Proof.* For \( n = 1 \), we have \( 1 = \frac{1(1+1)}{2} \) which is true.

Suppose (the induction hypothesis) that the statement (1) is true for \( n: \)

\[ 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}. \]
Then
\[
1 + 2 + 3 + \ldots + n + (n + 1) = \frac{n(n + 1)}{2} + (n + 1)
\]
\[
= \frac{n^2 + n + 2n + 2}{2}
\]
\[
= \frac{n^2 + 3n + 2}{2}
\]
\[
= \frac{(n + 1)(n + 2)}{2},
\]
which proves the statement (1) for \(n + 1\). By induction, the statement (1) is true for all natural numbers \(n\).

For the base case of induction, it is not necessary to use \(n = 1\). Any other base number \(k\) will work, and the result of induction will be that the statement is true for any \(n \geq k\).

There is also a technique called \textit{strong induction}, in which the inductive hypothesis is that the statement is true for 1, 2, 3,\ldots, \(n\).

### Problems

1. Prove that \(n! > 2^n\) for all \(n \geq 4\).

   \textit{Solution.} When \(n = 4\),
   \[
   4! = 24 > 16 = 2^4,
   \]
   so the statement is true. Assume \(n! > 2^n\). Then
   \[
   (n + 1)! = (n + 1)n! > (n + 1)2^n > 2 \cdot 2^n = 2^{n+1}
   \]
   (where we use the fact that \(n + 1 > 2\)). By induction, \(n! > 2^n\) for all \(n \geq 4\).

2. Prove that for any integer \(n \geq 1\), \(2^{2n} - 1\) is divisible by 3.

3. Prove that all numbers in the sequence 1007, 10017, 100117, 1001117, 10011117,\ldots are divisible by 53.

   \textit{Solution.} Let \(a_n = 100111 \cdots 117\) where there are \(n\) 1’s. Check \(a_0 = 1007 = 53 \cdot 19\). Now suppose \(a_n\) is divisible by 53. Generally, \(a_{n+1} = ((a_n - 7) + 1) \cdot 10 + 7 = 10a_n - 53\). Since both 53 and \(a_n\) are divisible by 53, so \(a_{n+1}\) is as well.
4. Let $F_k$ be the Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for $k > 1$. Show that:

$$\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$$

5. Let $r$ be a number such that $r + 1/r$ is an integer. Prove that for every positive integer $n$, $r^n + 1/r^n$ is an integer.

6. Prove that any square can be dissected into $n$ smaller squares (possibly of differing sizes) for every $n \geq 6$.

Solution. First, if you can dissect a square into $n$ squares, then you can dissect into $n + 3$ squares as follows: Choose any square in the dissection, and replace it with four squares, each one quarter of the original square. Since a square can be dissected into one square, induction proves that a square can be dissected into $1, 4, 7, 10, 13, \ldots$ squares. A square can be dissected into six squares as follows:

By induction, then, a square can be dissected into $6, 9, 12, 15, 18, \ldots$ squares. Finally, a square can be dissected into eight squares as follows:

By induction, a square can be dissected into $8, 11, 14, 17, \ldots$. In summary a square can be dissected into $1, 4, 6, 7, 8, 9, 10, \ldots$ squares, a list which includes every number greater than or equal to six.

7. Show that:

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}} = 2 \cos \left( \frac{\pi}{2^{n+1}} \right),$$

where there are $n$ 2s in the expression on the left.

8. If each person, in a group of $n$ people, is a friend of at least half the people in the group, then it is possible to seat the $n$ people in a circle so that everyone sits next to friends only.
9. Prove Bernoulli’s Inequality:

\[(1 + x)^n \geq 1 + nx\]

for every real number \(x \geq -1\) and every natural number \(n\).

10. Prove that \(2^n + 3^n + 5^n\) is divisible by 19 for all positive integers \(n\).

11. Prove that \(n^5/5 + n^4/2 + n^3/3 - n/30\) is an integer for \(n = 0, 1, 2, \ldots\)

12. You have coins \(C_1, C_2, \ldots, C_n\). For each \(k\), \(C_k\) is biased so that, when tossed, it has probability \(1/(2k + 1)\) of falling heads. If the \(n\) coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of \(n\).