Mathematical Induction

The natural numbers are the counting numbers: $1, 2, 3, 4, \ldots$. Mathematical induction is a technique for proving a statement - a theorem, or a formula - that is asserted about every natural number. For example,

$$1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}.$$  

This asserts that the sum of consecutive numbers from 1 to $n$ is given by the formula on the right. We want to prove that this will be true for $n = 1, n = 2, n = 3$, and so on. Now we can test the formula for any given number, say $n = 3$:

$$1 + 2 + 3 = \frac{3 \times 4}{2} = 6,$$

which is true. It is also true for $n = 4$:

$$1 + 2 + 3 + 4 = \frac{4 \times 5}{2} = 10.$$  

But how are we to prove this rule for every value of $n$? The method of proof is the following:

Principle of Mathematical Induction.

Suppose

1) (The base case) The statement is true for $n = 1$;

2) If the statement is true for $n$, then it is also true for $n + 1$;

Then the statement is true for every natural number $n$.

When the statement is true for $n = 1$, then according to 2), it will also be true for $n = 2$. But that implies it will be true for $n = 3$; which implies it will be true for $n = 4$. And so on. It will be true for every natural number. To prove a statement by induction, then, we must prove parts 1) and 2) above.

The hypothesis of part 2) - "The statement is true for $n$" - is called the inductive assumption, or the inductive hypothesis. It is what we assume when we prove a theorem by induction.

Example 1. Show that

$$1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}. \quad (1)$$  

Proof. For $n = 1$, we have $1 = \frac{1(1+1)}{2}$ which is true.

Suppose (the induction hypothesis) that the statement (1) is true for $n$:

$$1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}.$$
Then
\[
1 + 2 + 3 + \ldots + n + (n + 1) = \frac{n(n + 1)}{2} + (n + 1)
\]
\[
= \frac{n^2 + n + 2n + 2}{2}
\]
\[
= \frac{n^2 + 3n + 2}{2}
\]
\[
= \frac{(n + 1)(n + 2)}{2},
\]
which proves the statement (1) for \( n + 1 \). By induction, the statement (1) is true for all natural numbers \( n \).

For the base case of induction, it is not necessary to use \( n = 1 \). Any other base number \( k \) will work, and the result of induction will be that the statement is true for any \( n \geq k \).

There is also a technique called strong induction, in which the inductive hypothesis is that the statement is true for 1, 2, 3, \ldots, \( n \).

**Problems**

1. Prove that \( n! > 2^n \) for all \( n \geq 4 \).
2. Prove that for any integer \( n \geq 1 \), \( 2^{2n} - 1 \) is divisible by 3.
3. Prove that all numbers in the sequence 1007, 10017, 100117, 1001117, 10011117, \ldots are divisible by 53.
4. Let \( F_k \) be the Fibonacci numbers defined by \( F_0 = 0 \), \( F_1 = 1 \), and \( F_k = F_{k-1} + F_{k-2} \) for \( k > 1 \). Show that:
   \[
   \sum_{i=0}^{n} F_i^2 = F_n F_{n+1}
   \]
5. Let \( r \) be a number such that \( r + 1/r \) is an integer. Prove that for every positive integer \( n \), \( r^n + 1/r^n \) is an integer.
6. Prove that any square can be dissected into \( n \) smaller squares (possibly of differing sizes) for every \( n \geq 6 \).
7. Show that:
   \[
   \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}} = 2 \cos \left( \frac{\pi}{2n+1} \right),
   \]
   where there are \( n \) 2s in the expression on the left.
8. If each person, in a group of \( n \) people, is a friend of at least half the people in the group, then it is possible to seat the \( n \) people in a circle so that everyone sits next to friends only.

9. Prove Bernoulli’s Inequality:

\[
(1 + x)^n \geq 1 + nx
\]

for every real number \( x \geq -1 \) and every natural number \( n \).

10. Prove that \( 2^{2n} + 3^{2n} + 5^{2n} \) is divisible by 19 for all positive integers \( n \).

11. Prove that \( n^5/5 + n^4/2 + n^3/3 - n/30 \) is an integer for \( n = 0, 1, 2, \ldots \).

12. You have coins \( C_1, C_2, \ldots, C_n \). For each \( k \), \( C_k \) is biased so that, when tossed, it has probability \( 1/(2k + 1) \) of falling heads. If the \( n \) coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of \( n \).