The Greatest Integer function.

Definition. For a real number $x$, denote by $\lfloor x \rfloor$ the largest integer less than or equal to $x$.

A couple of trivial facts about $\lfloor x \rfloor$:

- $\lfloor x \rfloor$ is the unique integer satisfying $x - 1 < \lfloor x \rfloor \leq x$.
- $\lfloor x \rfloor = x$ if and only if $x$ is an integer.
- Any real number $x$ can be written as $x = \lfloor x \rfloor + \theta$, where $0 \leq \theta < 1$.

Some basic properties, with proofs left to the reader:

Proposition 1. For $x$ a real number and $n$ an integer:

1. $\lfloor x + n \rfloor = \lfloor x \rfloor + n$.
2. $\lfloor -x \rfloor = \begin{cases} -\lfloor x \rfloor & \text{if } x = \lfloor x \rfloor, \\ -\lfloor x \rfloor - 1 & \text{if } x \neq \lfloor x \rfloor. \end{cases}$
3. $\lfloor x/n \rfloor = \lfloor \lfloor x \rfloor /n \rfloor$ if $n \geq 1$.
4. $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$. More generally,

$$\lfloor nx \rfloor = \sum_{k=0}^{n-1} x + \frac{k}{n}.$$

The Legendre formula gives the factorization of $n!$ into primes:

Theorem 1 (Legendre Formula). For $n$ a positive integer,

$$n! = \prod_{pprime,p \leq n} p^{\alpha(p)}$$

where

$$\alpha(p) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

Note that the sum for $\alpha(p)$ is finite, and that $\alpha(p)$ is the highest power of $p$ that divides $n!$.

Proof. Among the first $n$ positive integers, those divisible by $p$ are $p, 2p, \ldots, tp$, where $t$ is the largest integer such that $tp \leq n$; in other words, $t$ is the largest integer less than or equal to $n/p$, so $t = \lfloor n/p \rfloor$. Thus there are exactly $\lfloor n/p \rfloor$ multiples of $p$ occurring in the product that defines $n!$, and they are

$$p, 2p, 3p, \ldots, \left\lfloor \frac{n}{p} \right\rfloor p.$$
With the same reasoning, the numbers between 1 and \( n \) which are divisible by \( p^2 \) are
\[
p^2, 2p^2, \ldots, \left\lfloor \frac{n}{p^2} \right\rfloor p^2
\]
and there are \( \left\lfloor n/p^2 \right\rfloor \) of these. Generally, \( \left\lfloor n/p^k \right\rfloor \) are divisible by \( p^k \) and so the total number of times \( p \) divides \( n! \) is
\[
\alpha(p) = \sum_{k=1}^{\infty} \left\lfloor n/p^k \right\rfloor.
\]

All of this material can be found in a good book on number theory, for example Burton, Elementary Number Theory. A deeper treatment is in Apostol, Introduction to Analytic Number Theory.

**Exercises**

1. Prove the statements in Proposition 1.
2. If \( 0 < y < 1 \), what are the possible values of \( \lfloor x \rfloor - \lfloor x - y \rfloor \)?

   *Solution.* Always 1 is possible, and also 0 unless \( x \) is an integer. \( \square \)

3. Let \( \{x\} = x - \lfloor x \rfloor \) denote the fractional part of \( x \). What are the possible values of \( \{x\} + \{-x\} \)?

   *Solution.* 0 if \( x \) is an integer, 1 otherwise. \( \square \)

4. Prove that \( \lfloor 2x \rfloor - 2 \lfloor x \rfloor \) is either 0 or 1.
5. Prove that \( \lfloor 2x \rfloor + \lfloor 2y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \).
6. For an integer \( n \geq 0 \), prove that \( \lfloor n/2 \rfloor - \lfloor -n/2 \rfloor = n \).
7. For an integer \( n \geq 1 \), the number of digits (in base ten) of \( n \) is \( 1 + \lfloor \log_{10}(n) \rfloor \).

**Problems**

1. How many zeros does the number 1000! end with?
Solution. One must find out how many factors of 10 are in 1000!. There will be more factors of 2 than 5, so compute the number of factors of five:

$$\left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{25} \right\rfloor + \left\lfloor \frac{1000}{125} \right\rfloor + \left\lfloor \frac{1000}{625} \right\rfloor = 200 + 40 + 8 + 1 = 249.$$  

There are 249 zeros at the end of 1000!.

2. If $n$ is a positive integer, prove that $\left\lfloor \sqrt{n} + \sqrt{n + 1} \right\rfloor = \left\lfloor \sqrt{4n + 2} \right\rfloor$.

Solution. First, $(\sqrt{n} + \sqrt{n + 1})^2 = 2n + 2\sqrt{n^2 + n} + 1$. Now, $n^2 < n^2 + n < (n + 1/2)^2$ so that $n < \sqrt{n^2 + n} < n + 1/2$. Then, 

$$\sqrt{4n + 1} < \sqrt{n + \sqrt{n + 1}} < \sqrt{4n + 2}.$$  

Squares are always odd or divisible by 4, so $4n + 2$ is never a square. Then $\left\lfloor \sqrt{4n + 1} \right\rfloor = \left\lfloor \sqrt{4n + 2} \right\rfloor$ and so $\left\lfloor \sqrt{n} + \sqrt{n + 1} \right\rfloor = \left\lfloor \sqrt{4n + 2} \right\rfloor$.  

3. Determine all positive integers $n$ such that $\left\lfloor \sqrt{n} \right\rfloor$ divides $n$.

Solution. When $n = k^2$, $n = k(k + 1)$, or $n = k(k + 2) = (k + 1)^2 - 1$.  

4. If $n$ is a positive integer, prove that 

$$\left\lfloor \frac{8n + 13}{25} \right\rfloor - \left\lfloor \frac{n - 12 - \frac{17}{25}}{3} \right\rfloor$$

is independent of $n$.

Solution. Bring the first term inside, get a common denominator. Check that it’s periodic mod 25, then check for $n = 0, \ldots, 24$.  

5. Prove that 

$$\sum_{k=1}^{n} \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor.$$  

Solution. Easy to do in two cases, $n$ even or $n$ odd, using $1 + \cdots + m = \frac{m(m+1)}{2}$.  

6. A sequence of real numbers is defined by the nonlinear first order recurrence 

$$u_{n+1} = u_n(u_n^2 - 3).$$  

(a) If $u_0 = 5/2$, give a simple formula for $u_n$.  
(b) If $u_0 = 4$, how many digits (in base ten) does $\lfloor u_{10} \rfloor$ have?
Solution. Prove by induction that if \( u_0 = \frac{d^2+1}{d} \) then \( u_n = \frac{d^{2n+1}}{d} \). In particular, for \( u_0 = 5/2, u_n = \frac{4^{n+1}}{2} \). For part (b), there are \( \left\lfloor 3^{10} \log_{10}(2 + \sqrt{3}) \right\rfloor + 1 \) digits. (MIT 18.S34 F’07).

7. Which positive integers can be written in the form \( n + \lfloor \sqrt{n} + 1/2 \rfloor \) for some positive integer \( n \)?

Solution. Looks like all but the squares. Haven’t proved it yet. (MIT 18.S34 F’07).

8. Prove that the sequence \( \left\lfloor (\sqrt{2})^n \right\rfloor \) contains infinitely many odd numbers.

Solution. \( (\sqrt{2})^n \) is even (a power of 2) when \( n \) is even. When \( n = 2k+1 \) is odd, \( (\sqrt{2})^n = 2^k \sqrt{2} \). Now \( 2^k \sqrt{2} \) is odd exactly when the \( k \)th binary digit of \( \sqrt{2} \) is 1. Since \( \sqrt{2} \) is irrational, its binary expansion must have infinitely many 1’s. (Original, inspired by the Graham-Pollak sequence).

9. Determine whether the improper integral

\[
\int_0^\infty (-1)^{\lfloor x^2 \rfloor} \, dx
\]

converges or diverges, where \( \lfloor \cdot \rfloor \) is the greatest integer function.

Solution. It converges. For \( \sqrt{n} \leq x < \sqrt{n+1} \), we have \( \lfloor x^2 \rfloor = n \), so that

\[
\int_0^\infty (-1)^{\lfloor x^2 \rfloor} \, dx = \sum_{n=0}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n}).
\]

Since the series is alternating and its terms approach zero, it converges. Source: Youngstown State Calculus Competition, 2005.

10. Let \( \{x\} \) denote the distance between the real number \( x \) and the nearest integer. For each positive integer \( n \), evaluate

\[
F_n = \sum_{m=1}^{6n-1} \min \left( \left\{ \frac{m}{6n} \right\}, \left\{ \frac{m}{3n} \right\} \right).
\]

(Here \( \min(a, b) \) denotes the minimum of \( a \) and \( b \).)

Solution. Putnam 1997 B1

11. Let \( a, b, c, d \) be real numbers such that \( \lfloor na \rfloor + \lfloor nb \rfloor = \lfloor nc \rfloor + \lfloor nd \rfloor \) for all positive integers \( n \). Prove that at least one of \( a + b, a - c, a - d \) is an integer.
12. Define a sequence $a_1 < a_2 < \cdots$ of positive integers as follows. Pick $a_1 = 1$. Once $a_1, \ldots, a_n$ have been chosen, let $a_{n+1}$ be the least positive integer not already chosen and not of the form $a_i + i$ for $1 \leq i \leq n$. Thus $a_1 + 1 = 2$ is not allowed, so $a_2 = 3$. Now $a_2 + 2 = 5$ is not allowed, so $a_3 = 4$. Then $a_3 + 3 = 7$ is not allowed, so $a_4 = 6$, etc. The sequence begins:

$$1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, \ldots$$

Find a simple formula for $a_n$. Your formula should enable you, for instance, to compute $a_{1000000}$.

**Solution.** (MIT 18.S34 F’07)

13. For a positive real number $\alpha$, define

$$S(\alpha) = \{ \lfloor n\alpha \rfloor : n = 1, 2, 3, \ldots \}.$$ 

Prove that $\{1, 2, 3, \ldots\}$ cannot be expressed as the disjoint union of three sets $S(\alpha), S(\beta), \text{ and } S(\gamma)$.

**Solution.** Very hard! (Putnam 1995 B6)