1. Show that every manifold has a nonzero complete vector field.

**Solution:** Let \( x \in M \) be any point, and choose a coordinate neighborhood \( U \) of \( x \) and a cutoff function \( f \) on \( M \) which is 1 at \( x \) and has compact support in \( U \). Let \( Y \) be any nonzero vector field defined on \( U \) (for example, \( Y = \frac{\partial}{\partial x^1} \), where \( x^1 \) is the first coordinate function). Define \( X = fY \) on \( U \), and extend to all of \( M \) by zero. Then \( X \) is a nonzero, smooth vector field on \( M \). Because \( X \) is compactly supported, it is complete.

2. Let \( f : M \to N \) be a smooth map of smooth manifolds, let \( \sigma : [a,b] \to M \) be a curve in \( M \), and let \( \omega \) be a one-form on \( N \). Show that

\[
\int_{\sigma} f^* \omega = \int_{f \circ \sigma} \omega.
\]

**Solution:**

\[
\int_{\sigma} f^* \omega = \int_a^b (f^* \omega)(\sigma'(t)) dt \\
= \int_a^b \omega(Tf \sigma'(t)) dt \\
= \int_a^b \omega((f \circ \sigma)'(t)) dt \\
= \int_{f \circ \sigma} \omega.
\]

3. Suppose \( \sigma \) is a locally conservative 1-form on \( S^2 \). Show there is \( f \in C^\infty(S^2) \) with \( \sigma = df \).

**Solution:** Since \( S^2 \) is simply connected, any closed curve \( \gamma \) on \( S^2 \) is homotopic to a point. Because \( \sigma \) is locally conservative, path integrals are homotopy invariant so that \( \int_\gamma \sigma = 0 \). Then \( \sigma \) is conservative, hence exact, hence equal to \( df \) for some \( f \). More specifically, one could fix \( x_0 \in M \) and define \( f(x) = \int_\gamma \sigma \) where \( \gamma \) is any curve joining \( x_0 \) to \( x \).

4. Let \( M \) be a manifold, and \( x, y \in M \). Show that for any \( D > 0 \), there is a Riemannian metric \( g \) on \( M \) with \( d_g(x, y) = D \).
5. Define a helix $H \subset \mathbb{R}^3$ parametrically by $(r \cos(\theta), r \sin(\theta), \theta)$ for $r \in [0, \infty]$ and $\theta \in \mathbb{R}$. Calculate the induced metric on $H$.

**Solution:** $dx = \cos(\theta)dr - r\sin(\theta)d\theta$, $dy = \sin(\theta)dr + r\cos(\theta)d\theta$, and $dz = d\theta$, so

$$dx^2 + dy^2 + dz^2 = dr^2 + (1 + r^2)d\theta^2$$

which gives the metric on $H$.

6. Given a Riemannian metric $g$ on the circle $S^1$, define the $L(g)$ to be the length (using $g$) of the curve that goes once around the circle. Show that any two metrics $g, h$ on $S^1$ with $L(g) = L(h)$ are isometric.

Hint: map to the canonical circle $C$ of length $L$, where $C = [0, L]/(L \sim 0)$ with metric $dt^2$.

**Solution:** Consider $S^1$ with parameter $\theta \in [0, 2\pi)$. Write $g = c(\theta)d\theta^2$ for some smooth function $c(\theta)$ which is periodic with period $2\pi$. Since $g$ is positive definite, $c$ is positive, so put $f(\theta) = \sqrt{c(\theta)}$, a positive, smooth periodic function with $g = f(\theta)^2d\theta^2$. Define

$$F(\theta) = \int_0^\theta f(t)dt.$$  

Then

$$F(\theta + 2\pi) - F(\theta) = \int_\theta^{\theta+2\pi} f(\theta)d\theta = \int_0^{2\pi} f(\theta)d\theta = \int_0^{2\pi} \sqrt{g(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta})}d\theta = L(g) = L$$

which shows that $\theta \rightarrow t = F(\theta)$ gives a well defined map from $S^1$ to $C$. Now

$$dt = dF(\theta) = F'(\theta)d\theta = f(\theta)d\theta$$

which shows that $F$ takes $g$ to $dt^2$, so that $(S^1, g)$ and $(C, dt^2)$ are isometric. Since $(S^1, g)$ and $(S^1, h)$ are both isometric to $(C, dt^2)$, they are isometric to each other.

7. Given a one form $\omega \in \mathcal{T}^1(M)$ and a vector field $X$, define $\mathcal{L}_X \omega$ by

$$\mathcal{L}_X \omega(Y) = \omega([X, Y]) - X(\omega(Y)).$$
Show that $\mathcal{L}_X\omega$ is a tensor.

**Solution:** For any vector field $Y$ and $f \in C^\infty(M)$, we have:

\[
\mathcal{L}_X\omega(Y) = \omega([X, fY]) - X\omega(fY) = \omega(X(fY) - (fY)X) - X(f\omega(Y))
\]

\[
= \omega((Xf)Y + fXY - fYX) - (Xf)\omega(Y) - fX\omega(Y)
\]

\[
= (Xf)\omega(Y) + f\omega([X,Y]) - (Xf)\omega(Y) - fX\omega(Y)
\]

\[
= f\mathcal{L}_X\omega(Y)
\]

Since $\mathcal{L}_X\omega$ is $C^\infty(M)$–linear, it’s a tensor.

8. Define an $r$-covariant tensor $\sigma$ on $\mathbb{R}^2$ by summing $2^r$ terms:

\[
\sigma = dx \otimes dx \otimes \cdots \otimes dx \otimes dx + dx \otimes dx \otimes \cdots \otimes dy \otimes dx + \cdots + dy \otimes dy \otimes \cdots \otimes dy \otimes dy
\]

where the sum is over all possible choices of $dx$ and $dy$ in each $r$-fold product term.

Find $\iota^*(\sigma)$, where $\iota$ is the inclusion map $S^1 \to \mathbb{R}^2$.

**Solution:** Replace $dx = \cos \theta d\theta$ and $dy = \sin \theta d\theta$, then use the binomial theorem to get $(\cos(\theta) + \sin(\theta))^r d\theta \otimes \cdots \otimes d\theta$. A similar approach is to notice that $\sigma = (dx + dy)^r$. 