1. Show that every manifold has a nonzero complete vector field.

2. Let $f : M \to N$ be a smooth map of smooth manifolds, let $\sigma : [a, b] \to M$ be a curve in $M$, and let $\omega$ be a one-form on $N$. Show that
$$\int_{\sigma} f^* \omega = \int_{f \circ \sigma} \omega.$$ 

3. Suppose $\sigma$ is a locally conservative 1-form on $S^2$. Show there is $f \in C^\infty(S^2)$ with $\sigma = df$.

4. Let $M$ be a manifold, and $x, y \in M$. Show that for any $D > 0$, there is a Riemannian metric $g$ on $M$ with $d_g(x, y) = D$.

5. Define a helix $H \subset \mathbb{R}^3$ parametrically by $(r \cos(\theta), r \sin(\theta), \theta)$ for $r \in [0, \infty]$ and $\theta \in \mathbb{R}$. Calculate the induced metric on $H$.

6. Given a Riemannian metric $g$ on the circle $S^1$, define the $L(g)$ to be the length (using $g$) of the curve that goes once around the circle. Show that any two metrics $g, h$ on $S^1$ with $L(g) = L(h)$ are isometric.

   Hint: map to the canonical circle $C$ of length $L$, where $C = [0, L]/(L \sim 0)$ with metric $dt^2$.

7. Given a one form $\omega \in \mathcal{T}^1(M)$ and a vector field $X$, define $\mathcal{L}_X \omega$ by
$$(\mathcal{L}_X \omega)(Y) = \omega([X, Y]) - X.\omega(Y).$$

   Show that $\mathcal{L}_X \omega$ is a tensor.

8. Define an $r$-covariant tensor $\sigma$ on $\mathbb{R}^2$ by summing $2^r$ terms:
$$\sigma = dx \otimes dx \otimes \cdots \otimes dx \otimes dx$$
$$+ dx \otimes dx \otimes \cdots \otimes dx \otimes dy$$
$$+ dx \otimes dx \otimes \cdots \otimes dy \otimes dx$$
$$\cdots$$
$$+ dy \otimes dy \otimes \cdots \otimes dy \otimes dy$$

   where the sum is over all possible choices of $dx$ and $dy$ in each $r$-fold product term.

   Find $\iota^*(\sigma)$, where $\iota$ is the inclusion map $S^1 \to \mathbb{R}^2$. 