1. For a one form $\omega \in T^1(M)$, suppose that, for all vector fields $X$, $\mathcal{L}_X\omega = 0$. Prove that the coefficients $\omega_i$ are constant functions in any coordinate system.

**Solution:** Generally, $(\mathcal{L}_X\omega)(Y) = X.\omega(Y) - \omega([X,Y])$. Let $x^1, \ldots, x^n$ be local coordinates on $M$, and write $\omega = \sum_i \omega_i dx^i$. Using $X = \frac{\partial}{\partial x^i}, Y = \frac{\partial}{\partial x^j}$, we have $[X,Y] = 0$ and

$$0 = (\mathcal{L}_X\omega)(Y) = X.\omega(Y) = X.\omega_j = \frac{\partial \omega_j}{\partial x^i}.$$ 

Since $\omega_j$ has all of its partial derivatives equal to zero, $\omega_j$ is constant.

2. Recall that $M$ is parallelizable if the tangent bundle $TM$ is trivial. Show that a parallelizable manifold is orientable. Give an example to show that an orientable manifold need not be parallelizable.

**Solution:** Parallelizable means $TM = M \times \mathbb{R}^n$, or equivalently there are $n$ non-vanishing vector fields on $M$ which form a basis for each tangent space.

The sphere $S^2$ is orientable but $TS^2$ is non-trivial - there is no non-vanishing vector field on $S^2$.

If $M$ is parallelizable, let $e_1, \ldots, e_n$ be a basis for $\mathbb{R}^n$ and let $e_1^*, \ldots, e_n^*$ be the dual basis. Then $\mu : M \to \bigwedge^n TM = M \times \bigwedge^n \mathbb{R}^n$ by $\mu(p) = (p, e_1^* \wedge \cdots \wedge e_n^*)$ is a non-vanishing $n$-form on $M$, so $M$ is oriented.

Another approach: let $X_1, \ldots, X_n$ be non-vanishing vector fields on $M$ which give a basis for $T_p(M)$ for all $p \in M$. Let $\omega_1, \ldots, \omega_n \in \bigwedge^1(M)$ give, at each point $p$, the basis of $T^*_p(M)$ dual to the $X_i$. Then $\mu = \omega_1 \wedge \cdots \wedge \omega_n$ is an $n$-form on $M$, and $\mu$ is nonvanishing since $\mu(X_1, \ldots, X_n) = 1$.

3. Let $\phi \in T^2(M)$ be a two-tensor, and define $\phi_\Delta(X) = \phi(X,X)$. Does $\phi_\Delta$ define a one-form?

**Solution:** No, unless $\phi_\Delta \equiv 0$.

Suppose $\phi_\Delta$ is a one-form. For any $X \in \mathfrak{X}(M)$ and any $C^\infty$ function $f$,

$$f \phi_\Delta(X) = \phi_\Delta(fX) = \phi(fX,fX) = f^2 \phi(X,X) = f^2 \phi_\Delta(X)$$

If $\phi_\Delta(X)$ is not identically zero, then there is $X$ so that $\phi_\Delta(X) \neq 0$ at some point $p \in M$. Then $f(p) = f(p)^2$ for any smooth function $f$ on $M$, including, for example $f(x) \equiv 2$. This is a contradiction.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and let $M$ be the graph of $f$ with the metric induced from $\mathbb{R}^3$. Show that the metric volume 2-form on $M$ is $\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \, dx \wedge dy$. 

Solution: With \( z = f(x, y) \)

\[
dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy
\]

Way 1: Using the local expression for the metric volume form (Boothby pg 219/Lee Prop. 9.21)

\[
dx^2 + dy^2 + dz^2 = \left( 1 + \left( \frac{\partial f}{\partial x} \right)^2 \right) dx^2 + \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} dx dy + \frac{\partial f}{\partial y} dy dx + \left( 1 + \left( \frac{\partial f}{\partial y} \right)^2 \right) dy^2
\]

The metric \( g_{ij} \) is given by the matrix

\[
\begin{pmatrix}
1 + \left( \frac{\partial f}{\partial x} \right)^2 & \frac{\partial f}{\partial y} \\
\frac{\partial f}{\partial y} & 1 + \left( \frac{\partial f}{\partial y} \right)^2
\end{pmatrix}
\]

Then \( g = \det(g_{ij}) = 1 + \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 \) and the volume form is \( \sqrt{g} dx \wedge dy \).

Way 2: Taking the cross product of the tangent vectors \((1, 0, \frac{\partial f}{\partial x})\) and \((0, 1, \frac{\partial f}{\partial y})\) gives a normal vector field

\[
N = -\frac{\partial f}{\partial x} \frac{\partial}{\partial x} - \frac{\partial f}{\partial y} \frac{\partial}{\partial y} + \frac{\partial}{\partial z}
\]

and unit normal \( n = \frac{N}{||N||} \). Then the volume form on \( M \) is given by

\[
\iota_n(dx \wedge dy \wedge dz) = \frac{1}{||N||} \left( -\frac{\partial f}{\partial x} dy \wedge dz + \frac{\partial f}{\partial y} dx \wedge dz + dx \wedge dy \right)
\]

\[
= \frac{1}{||N||} \left( -\left( \frac{\partial f}{\partial x} \right)^2 dy \wedge dx + \left( \frac{\partial f}{\partial y} \right)^2 dx \wedge dy + dx \wedge dy \right)
\]

\[
= \sqrt{1 + \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2} dx \wedge dy.
\]

5. Let \( \gamma : [0, 2\pi] \to \mathbb{R}^3 \) be the ‘slinky curve’

\[
\gamma(t) = \left( \cos(t)(\cos(10t) + 2), \sin(t)(\cos(10t) + 2), \frac{2t}{3} + \sin(10t) \right)
\]

and let \( \alpha = ydx + xdy + dz \). Compute \( \int_\gamma \alpha \).

Solution: Note that \( d\alpha = dy \wedge dx + dx \wedge dy = 0 \), so \( \alpha \) is closed and therefore exact. There is some \( f(x, y, z) \) with \( df = \alpha \). Since \( \frac{\partial f}{\partial x} = 1 \), \( f(x, y, z) = g(x, y) + z \). Since \( \frac{\partial f}{\partial y} = y \), \( g(x, y) = xy + h(y) \), so \( f = xy + z + h(y) \). Finally, \( \frac{\partial f}{\partial y} = x \), so \( h(y) = c \) for some constant \( c \) (which we set to zero) so that \( f(x, y, z) = xy + z \).
Now \( \gamma(0) = (3, 0, 0) \) and \( \gamma(2\pi) = (3, 0, 4\pi/3) \), so that
\[
\int_{\gamma} \alpha = \int_{\gamma} df = f(3, 0, 4\pi/3) - f(3, 0, 0) = 4\pi/3
\]

6. Let \( D^n \) and \( S^{n-1} \) be the unit ball and unit sphere in \( \mathbb{R}^n \).
   (a) With
   \[
   \mu = \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge dx^2 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^n,
   \]
   show that \( \mu|_{S^{n-1}} \) is the volume form on \( S^{n-1} \).
   (b) Show that \( \text{vol} S^{n-1} = n \text{vol} D^n \). Hint: Apply Stokes’ theorem to \( \mu \) integrated over \( S^{n-1} \).

**Solution:**
   (a) The unit normal field to \( S^{n-1} \) is given by
   \[
   N = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i},
   \]
   and
   \[
   \iota_N dx^1 \wedge \cdots \wedge dx^n = \mu
   \]
   (b) Compute \( d\mu = ndx^1 \wedge \cdots \wedge dx^n \). Then applying Stokes’ Theorem with \( S^{n-1} = \partial D^n \),
   \[
   \text{vol} S^{n-1} = \int_{S^{n-1}} \mu = \int_{D^n} d\mu = \int_{D^n} ndx^1 \wedge \cdots \wedge dx^n = n \text{vol} D^n.
   \]

7. Let \( (M,g) \) be a Riemannian manifold with metric two-form \( g \). A vector field \( X \in \mathfrak{X}(M) \) is called a Killing field if \( \mathcal{L}_X g = 0 \). Let \( X \) be a complete Killing field with flow \( \phi_t : M \to M \).

This problem is to show that \( X \) preserves lengths of curves on \( M \) (and therefore preserves distances between points of \( M \)).

Suppose \( \gamma : [a,b] \to M \) is a smooth curve. Then \( \phi_t \circ \gamma \) is also a smooth curve. Show that, for all \( t \),
\[
\text{len}(\phi_t \circ \gamma) = \text{len}(\gamma)
\]

Hint: Show that the \( t \) derivative of the length vanishes.
Solution:

\[
\frac{\partial}{\partial t} \text{len}(\phi_t \circ \gamma) = \frac{\partial}{\partial t} \int_a^b \sqrt{g((\phi_t \circ \gamma)'(s), (\phi_t \circ \gamma)'(s))} ds
\]

\(= \int_a^b \frac{\partial}{\partial t} \sqrt{g(T\phi_t \gamma'(s), T\phi_t \gamma'(s))} ds\)

\(= \int_a^b \frac{\partial}{\partial t} \sqrt{(\phi_t^* g)(\gamma'(s), \gamma'(s))} ds\)

\(= \int_a^b \frac{1}{2\sqrt{(\phi_t^* g)(\gamma'(s), \gamma'(s))}} \frac{\partial}{\partial t} (\phi_t^* g)(\gamma'(s), \gamma'(s)) ds\)

\(= \int_a^b \frac{1}{2\sqrt{(\phi_t^* g)(\gamma'(s), \gamma'(s))}} (L_{\gamma'} g)(\gamma'(s), \gamma'(s)) ds\)

\(= 0\)

So the function \(\text{len}(\phi_t \circ \gamma)\) has vanishing \(t\) derivative, and is constant in \(t\).

8. Let \(\alpha, \beta\) be \(k\)-forms on a smooth \(n\)-manifold \(M\). Let \(S\) be a \(k\)-dimensional submanifold of \(M\) (without boundary). If \([\alpha] = [\beta] \in H^k(M)\), (i.e. \(\alpha\) and \(\beta\) represent the same cohomology class) then show that

\[
\int_S \alpha = \int_S \beta
\]

**Solution:** Write \(\alpha = \beta + d\tau\) for some \(k-1\) form \(\tau\). Then applying Stokes’ Theorem,

\[
\int_S \alpha = \int_S \beta + \int_S d\tau = \int_S \beta + \int_{\partial S} \tau = \int_S \beta
\]

Here are some reasonable questions from our texts, that were not already assigned.

From Lee: Ch1 Problem 19. Ch 2 Exercises 2.82, 2.90, 2.100, 2.111, 2.117. Ch2 Problems 3, 4, 23, 24, 30. Ch6 Problem 1. Ch 7 Problem 6. Ch 8 Exercises 8.11, 8.18, 8.52, 8.70, 8.71, 8.72, 8.77, 8.80. Ch 8 Problems 2.4, 6.9, 10.11, 12. Ch 9 Exercises 9.58, 9.59. Ch 9 Problem 1. Ch 10 Exercises 10.1, 10.2. Ch 10 Problems 1, 5, 7, 8, 9.

From Boothby (Chapter V): Section 1 # 3, 7. Section 3 # 1, 4, 7. Section 5 # 2, 6. Section 6 # 1, 2, 3, 4, 7. Section 8 # 3, 5, 6, 7, 8.