Questions get two ratings: A number which is relevance to the course material, a measure of how much I expect you to be prepared to do such a problem on the exam. 3 means ‘of course you know this information’, 1 means ‘you probably need to check something in the book for this one’. Given that you know the material, the starred problems are harder.


1. (3) Show that a connected manifold is path connected.

Solution: Pick $p \in M$. Let $C = \{x \in M | \text{There is a path between } x \text{ and } p\}$.

If $x \in C$, choose a chart $(U, \varphi)$ containing $x$ and a path $c$ from $p$ to $x$. Since $\varphi(U)$ is open in $\mathbb{R}^m$, there is $r > 0$ with the open ball $B = B(\varphi(x), r) \subset \varphi(U)$. For any $y \in \varphi^{-1}(B)$, there is a path $s$ from $\varphi(x)$ to $\varphi(y)$ in $B$, so that $\varphi^{-1} \circ s$ is a path in $M$ from $x$ to $y$. Then $c$ followed by $\varphi^{-1} \circ s$ is a path in $M$ from $p$ to $y$, so that $y \in C$. That is, $C$ contains the open neighborhood $\varphi^{-1}(B)$ of $x$, and so $C$ is open.

2. (2) Let $D$ be a derivation on $C^\infty(M)$. Suppose $f, g \in C^\infty(M)$, and that $g$ is never 0. Prove the quotient rule:

$$D \left( \frac{f}{g} \right) = \frac{gDf - fDg}{g^2}$$

Solution: Since $g \neq 0$, $f/g \in C^\infty(M)$. Then by Leibniz’ rule:

$$Df = D \left( g \cdot \frac{f}{g} \right) = Dg \cdot \frac{f}{g} + gD \left( \frac{f}{g} \right)$$

Solving for $D(f/g)$ gives the result.

3. (3) Given a sequence of open sets $\{U_i\}_{i=1}^\infty$ with $\overline{U}_n \subset U_{n+1}$ for all $n$, and with $\bigcup_{i=1}^\infty U_n = M$. Say that a sequence $x_1, x_2, \ldots$ leaves all $U$ if for any $n$ there is $N$ so that $x_i \notin U_n$ for $i > N$.

Show that there is a smooth function $f : M \to \mathbb{R}$ so that $\lim_{i \to \infty} f(x_i) = +\infty$ for any sequence $\{x_i\}_{i=1}^\infty$ which leaves all $U$.

Solution: Let $b_n$ be a cutoff function which is 1 on $U_{n-1}$ and 0 on the complement of $U_n$ (for $n = 1$, set $b_1 = 0$). Let $\phi_n = 1 - b_n$, so $\phi_n$ is 0 on $U_{n-1}$ and 1 outside of $U_n$. For $x \in U_n$, there is a neighborhood $V \subset U_n$ of $x$, and for any $i > n$, $\phi_i \equiv 0$ on $V$. Define $f = \sum_{i=1}^\infty \phi_i$, which is a finite sum in a neighborhood of any $x$, so $f$ is smooth.

Suppose a sequence $\{x_i\}$ leaves all $U$. Given $n > 0$, there is $N$ so that $x_i \notin U_n$ for $i > N$. Then for $i > N$, $x_i \notin U_n$ so

$$f(x_i) \geq \sum_{k=1}^n \phi_k(x) = \sum_{k=1}^n 1 = n$$

which shows $f(x_i) \to \infty$. 
4. Which of these homeomorphisms are diffeomorphisms from \( \mathbb{R}^2 \to \mathbb{R}^2 \)?

(a) \((x, y) \to (x^3, y^3)\)  
(b) \((x, y) \to (x^3 + x, y^3 + y)\)  
(c) \((x, y) \to (x \cos(x^2 + y^2) - y \sin(x^2 + y^2), x \sin(x^2 + y^2) + y \cos(x^2 + y^2))\)

**Solution:** Parts \(b,c\) are diffeos but \(a\) is not. Part \(c\) rotates \((x, y)\) by the angle \(r^2 = x^2 + y^2\).

**(**2) 5. Let \(M(2)\) denote the space of \(2 \times 2\) matrices with real entries. Let \(N = \{A \in M(2) | A \neq 0, \det(A) = 0\}\). Show that \(N\) is a manifold.

**Solution:**

Way 1: Let \(U_\ell\) be the set of matrices in \(N\) with nonzero left column, and \(U_r\) be the set of matrices in \(N\) with nonzero right column. Note that \(N = U_\ell \cup U_r\). For \(A \in U_\ell\), write \(A = \begin{pmatrix} x & \lambda x \\ y & \lambda y \end{pmatrix}\) (which we can do because the columns of \(A\) are linearly dependent). Put \(\phi_\ell(A) = (x, y, \lambda)\). Similarly, for \(A \in U_r\), write \(A = \begin{pmatrix} \lambda x & x \\ \lambda y & y \end{pmatrix}\) and put \(\phi_r(A) = (x, y, \lambda)\). On \(U_\ell \cap U_r\), the change of coordinates map is given by \((\phi^{-1}_r \circ \phi_\ell)(x, y, \lambda) = (\lambda x, \lambda y, \lambda^{-1})\), which is smooth. The inverse \(\phi^{-1}_r \circ \phi_\ell\) has the same formula and is also smooth. Then \((U_\ell, \varphi_\ell)\) and \((U_r, \varphi_r)\) define an atlas on \(N\).

Way 2: For \(A \in N\), the kernel of \(A\) is a line through the origin. Let \(U_h\) be the set of \(A \in N\) whose kernel is not horizontal, and \(U_v\) be the \(A\) with kernel which is not vertical. For \(A \in U_h\), let \(\theta \in (0, \pi)\) be the angle that \(\ker A\) makes with the positive \(x\)-axis (well defined on \(U_h\)). Let \(R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}\), clockwise rotation by \(\theta\). Then \(A R_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0\), so \(A R_\theta = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}\) and define \(\varphi_h(A) = (x, y, \theta)\). Note \(\begin{pmatrix} x \\ y \end{pmatrix} = A R_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}\). Similarly define \(\varphi_v(A)\) on \(U_v\), except \(\theta \in (-\pi/2, \pi/2)\). When \(\ker A\) has positive slope, \(\varphi_h(A) = \varphi_v(A)\) so the coordinate change is just the identity. When \(\ker A\) has negative slope, if \(\varphi_h(A) = (x, y, \theta)\) then \(\varphi_v(A) = (-x, -y, \theta - \pi)\) since \(R_{\theta - \pi} = -R_\theta\). Then \((U_h, \varphi_h)\) and \((U_v, \varphi_v)\) define an atlas on \(N\).

Note: Way 1 and way 2 are reminiscent of putting stereographic and angular coordinates on a circle, respectively. In both cases, it’s easy to see that the set of matrices in \(N\) with a fixed \(\lambda\) or \(\theta\) form a two dimensional vector space, so that \(N\) is a vector bundle over the circle. \(N\) is a trivial bundle over \(S^1\) (show it!) so that \(N\) is diffeomorphic to \(\mathbb{R}^2 \times S^1\).

Bonus: Generalize these results to \(N \subset M(n)\), the set of \(n \times n\) matrices with one dimensional kernel. What dimension is \(N\)? Generally, \(N\) is a bundle over \(\mathbb{R}P^{n-1}\) with projection \(\pi : N \to \mathbb{R}P^{n-1}\) given by \(\pi(A) = \ker A\). Is this a trivial bundle?
6. For a smooth map of manifolds \( f : M \to N \), say that \( f \) is **self-transverse** if for all \( x, y \in M \) there are neighborhoods \( x \in U, y \in V \) so that \( f|_U \cap f|_V \).

(a) Give an example of \( M, N \) and \( f : M \to N \) which is not self-transverse.

(b) Give an example of \( M, N \) and \( f : M \to N \) which is self-transverse and not injective.

(c) Suppose \( f : M \to N \) is a self-transverse immersion. Show \( K = \{ x \in M | \exists x' \in M \text{ with } f(x) = f(x') \} \) is a regular submanifold of \( M \).

Except that part (c) is false! (*) Give an example to show part (c) is false.

**Solution:**

(a) Here are some: \( f : \mathbb{R} \to \mathbb{R}^2 \) by \( f(t) = (\cos t, \sin t) \) is not self-transverse, for example between \( t = 0 \) and \( t = 2\pi \). Any curve in \( \mathbb{R}^3 \) which intersects itself is not self-transverse.

If \( M \) is the disjoint union of two lines, \( f : M \to \mathbb{R}^2 \) by \( f(s) = (s, 0) \) and \( f(t) = (t, t^2) \) is not self-transverse.

(b) \( f \) could map a disjoint union of two lines onto the two axes in \( \mathbb{R}^2 \). Or, let \( f(t) = (t \cos(t), t \sin(t)) \), a spiral whose \( t > 0 \) branch has transverse intersection with its \( t < 0 \) branch.

(c) Let \( M \) be the disjoint union of three copies of \( \mathbb{R}^2 \) and map \( M \) to the three coordinate planes in \( \mathbb{R}^3 \). Then \( M \) is self-transverse, but in each copy of \( \mathbb{R}^2 \), \( K \) is the union of the coordinate axes, which is not a manifold.

7. (*) Let \( M \) be a regular submanifold of \( N \), and let \( X \) be a vector field on \( M \). Show there is a vector field \( \tilde{X} \) on \( N \) with \( \tilde{X}|_M = X \).

**Solution:** For \( p \in M \), let \( (x_1, \ldots, x_n) \) be single slice coordinates on an open set \( U \subset N \) with \( p \in U \). So \( M \cap U = \{ (x_1, \ldots, x_m, 0, \ldots, 0) \} \cap U \). On \( M \cap U \), write \( X = \sum_{i=1}^m X_i(x_1, \ldots, x_m) \frac{\partial}{\partial x_i} \). Define a vector field on \( U \) by

\[
\tilde{X}_U(x_1, \ldots, x_n) = \sum_{i=1}^m X_i(x_1, \ldots, x_m) \frac{\partial}{\partial x_i}
\]

so that \( \tilde{X}_U|_M = X \).

Let \( V = N - M \), and define \( \tilde{X}_V = 0 \). Now \( V \) and the collection of \( U \) as above are an open cover for \( M \). Take a locally finite refinement of this cover, say \( \{W_\alpha\} \). Each \( W_\alpha \) is a subset of some \( U \) (or \( V \)), so each has a vector field \( \tilde{X}_\alpha = \tilde{X}_U|_{W_\alpha} \). Let \( \{\varphi_\alpha\} \) be a partition of unity subordinate to \( \{W_\alpha\} \). Define \( \tilde{X} = \sum_\alpha \varphi_\alpha \tilde{X}_\alpha \). Fix \( p \in M \), if \( p \in W_\alpha \) for some \( \alpha \), then \( X_\alpha(p) = X(p) \). Therefore

\[
\tilde{X}(p) = \sum_{\alpha, p \in W_\alpha} \varphi_\alpha(p) \tilde{X}_\alpha(p) = \left( \sum_{\alpha, p \in W_\alpha} \varphi_\alpha(p) \right) X(p) = X(p).
\]
8. Show that the set of closed disks in \( \mathbb{R}^2 \) which don’t contain the origin is a manifold, and show it is diffeomorphic to \( S^1 \times \mathbb{R}^2 \).

**Solution:** We can parameterize the set of closed disks by one chart with domain \( H = \{ (x, y, z) \in \mathbb{R}^3 | z > 0 \} \), by sending \( (x, y, z) \in H \) to a disk with center \( (x, y) \) and radius \( z \). Those which don’t contain the origin form a manifold because they correspond to the open set \( V = \{ (x, y, z) | x^2 + y^2 > z^2 \} \subset H \).

Given \( (e^{i\theta}, a, b) \in S^1 \times \mathbb{R}^2 \), define \( f(e^{i\theta}, a, b) = (e^{a} + e^{b}) \cos(\theta), (e^{a} + e^{b}) \sin(\theta), e^{b} = (x, y, z) \in V \)

This map is well defined since adding \( 2\pi \) to \( \theta \) has no effect on \( (x, y, z) \). \( f \) is smooth, one-to-one onto \( V \), and \( f^{-1}(x, y, z) = \left( \frac{x + iy}{\sqrt{x^2 + y^2}}, \log(\sqrt{x^2 + y^2} - z), \log z \right) \) is also smooth.

9. Let \( \sigma \) be a curve (embedded 1-manifold) in \( \mathbb{R}^3 \), and let \( \sigma_a \) be the rescaled image of \( \sigma \) under the map \( (x, y, z) \rightarrow (ax, ay, az) \), for some \( a > 0 \). For \( p \in \sigma \), compute the curvature of \( \sigma_a \) at \( ap \) in terms of \( a \) and the curvature of \( \sigma \) at \( p \).

**Solution:** Let \( \sigma(t) \) be a unit speed parameterization with \( \sigma(0) = p \). Then \( \sigma_a(t) = a\sigma(t/a) \) is a unit speed parameterization of \( \sigma_a \) with \( \sigma_a(0) = ap \). Compute the unit tangent vector and it’s derivative as:

\[
\sigma'_a(t) = \sigma'(t/a) \\
T_a(t) = T(t/a) \\
T'_a(t) = \frac{1}{a} T'(t/a)
\]

Since both curves are unit speed, the curvature satisfies \( \kappa_a(ap) = \frac{1}{a} \kappa(p) \).

10. Suppose \( M \) is an embedded surface in \( \mathbb{R}^3 \), and let \( N \) be the rescaled image of \( M \) under the map \( (x, y, z) \rightarrow (ax, ay, az) \), for some \( a > 0 \). Compute the Gauss curvature \( K_N(ap) \) of \( N \) at \( ap \) in terms of \( a \) and the Gauss curvature \( K_M(p) \) of \( M \) at \( p \).

**Solution:** Let \( \sigma(t) \) be a unit speed curve in \( N \) with \( \sigma(0) = ap \). Put \( \tau(t) = \frac{1}{a}\sigma(at) \), a curve in \( M \). Notice \( \tau'(t) = \frac{1}{a} \sigma'(at) \cdot a = \sigma'(at) \), so \( \tau \) also has unit speed, and \( \tau'(0) = \sigma'(0) \). This shows the tangent planes \( T_pM \) and \( T_{ap}N \) are parallel, so a unit normal vector for \( M \) at \( p \) is also a unit normal vector for \( N \) at \( ap \). Let \( n \) be a unit normal field on \( M \) and also for \( N \), which means \( n(ap) = n(p) \).
Now compute the shape operator $S_N$ on $N$ in terms of $S_M$ on $M$:

\[
S_N(\sigma'(0)) = (n \circ \sigma)'(0) = \left. \frac{d}{dt} n(a\tau(t/a)) \right|_{t=0} = (n(\tau(t/a)))'_{t=0} = (n \circ \tau)'(0) \cdot \frac{1}{a} = \frac{1}{a} S_M(\tau'(0)) = \frac{1}{a} S_M(\sigma'(0)).
\]

So $S_N = \frac{1}{a} S_M$ and, taking determinants, $K_N(ap) = \frac{1}{a^2} K_M(p)$. Note that this checks with the situation where $M$ is a sphere of radius 1, where $K_M \equiv 1$, and $N$ is a sphere of radius $a$ with $K_N \equiv \frac{1}{a^2}$.

It is also possible to do this by showing that curvature scales by $\frac{1}{a}$ for curves, and since Gauss curvature is the product of the two principal curvatures it must scale by $\frac{1}{a^2}$.

(1*) Let $c = c(s)$ be a unit speed curve in $\mathbb{R}^3$, and suppose the Frenet frame $T, N, B$ is defined for all $s$. Define $f : \mathbb{R}^2 \to \mathbb{R}^3$ by $f(s, t) = c(s) + tN(s)$. Notice that for fixed $s$, $f(s, t)$ is the normal line to the curve at $c(s)$, and for fixed $t$, $f(s, t)$ is a curve ‘parallel’ to $c$ at distance $t$.

Find all points where $f$ fails to be an immersion.

In the case where $c$ is a planar curve, $f : \mathbb{R}^2 \to \mathbb{R}^2$ and these points are the critical values of $f$.

**Solution:** Let $\kappa$ and $\tau$ be the curvature and torsion of $c$, and recall $N' = -\kappa T + \tau B$.

\[
\frac{\partial f}{\partial s} = c' + tN' = T - t\kappa T + t\tau B = (1 - t\kappa)T + t\tau B.
\]

\[
\frac{\partial f}{\partial t} = N.
\]

$f$ is an immersion except when these two vectors are dependent, which we can check with the cross product:

\[
\frac{\partial f}{\partial s} \times \frac{\partial f}{\partial t} = (1 - t\kappa)B - t\tau T.
\]

Since $B$ and $T$ are independent, this vanishes when $t\tau = 0$ and $1 - t\kappa = 0$. Since $t\kappa = 1$, neither $t$ nor $\kappa$ can vanish. Therefore, $f$ is an immersion except when both $\tau(s) = 0$ and $t = \frac{1}{\kappa(s)}$.

Additional remark: Geometrically, $\tau = 0$ means that $c$ is planar to 3rd order at $p = c(s)$. Normally a curve is planar only to 2nd order – see Lee, Exercise 4.7 for a Taylor expansion that shows this. The critical value is then in the plane of the curve, at $\frac{1}{\kappa}$ along the normal line from $p$. This is the center of curvature for the curve at $p$, which is the center of a circle (radius $\frac{1}{\kappa}$) that is tangent to the curve at $p$ to order 2. When $c$ is a plane curve, the set of critical values of $f$ is known as the evolute of $c$. The Wikipedia page for evolute has a pretty animation of $f$ as $s$ varies.
(2) 12. Let $M(2)$ denote the vector space of $2 \times 2$ matrices. Since $M(2)$ is a vector space, the tangent space to $M(2)$ at the identity is naturally identified with $M(2)$. Let $SL(2) \subset M(2)$ be the set of matrices of with determinant 1.

(a) Show that $SL(2)$ is a manifold.
(b) What is dim $SL(2)$?
(c) * Show that the tangent space at the identity, $T_I SL(2)$, is exactly the space of traceless matrices $\{A \in M(2) | \text{tr}(A) = 0\}$.

Bonus: Do this problem for $n \times n$ matrices instead of $2 \times 2$.

Solution: For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, det $A = ad - bc$. Then $T \text{det} = (d, -c, -b, a)$ which has rank 1 unless $A = 0$. So any value other than 0 is a regular value for det. In particular, 1 is a regular value for det, so $SL(2)$, the set of matrices with determinant 1, is a manifold. Because dim $M(2) = 4$ and det has rank 1, dim $SL(2) = 3$.

Let $X$ be a tangent vector to $SL(2)$ at the identity. Represent $X$ by a curve $C(t) = \begin{pmatrix} a(t) \\ b(t) \\ c(t) \\ d(t) \end{pmatrix} \in SL(2)$, with $C(0) = I$ and $C'(0) = X$. We know det $C(t) = 1$, so take the derivative of both sides to get

$$\frac{d}{dt}((a(t)d(t) - b(t)c(t)) \bigg|_{t=0} = 0,$$

so

$$a'(0)d(0) + a(0)d'(0) - b'(0)c(0) - b(0)c'(0) = 0.$$ (10)

Now $C(0) = I$, so $b(0) = c(0) = 0$ and $a(0) = d(0) = 1$, so we get $a'(0) + d'(0) = 0$, or that $0 = \text{tr} C''(0) = \text{tr} X$.

Although one can generalize the argument above for $n > 2$ using the combinatorial definition of determinant as a sum over permutations, an easier approach is to write det $C(t)$ as a product of the eigenvalues of $C(t)$.

(*1) 13. Suppose $M \subset \mathbb{R}^3$ is a surface, and assume that for any closed curve $C : S^1 \to M$ there is a continuous unit normal field to $M$ defined along $C$. Show that $M$ is orientable.

Solution: Assume $M$ is connected. If not, orient each component of $M$ separately. Let $p_0 \in M$, and let $N_0$ be a unit normal to $M$ at $p_0$. For $p \in M$, choose any smooth curve $\sigma$ joining $p_0$ with $p$, and extend $N_0$ along $\sigma$ to a unit normal vector $N_p$. We need to show $N_p$ is well defined. Suppose $\tau$ is any other curve joining $p_0$ with $p$. Together, $\sigma$ and $\tau$ form a closed curve $C$, so there is a continuous unit normal field $V$ along $C$, and by replacing with $-V$ if necessary, we may assume $V = N_0$ at $p_0$. Then the extension of $N_0$ along $\sigma$ agrees with $V$ at $p$, and so does the extension of $N_0$ along $\tau$, so $N_p$ is well defined. Then $N$ is a unit normal field on $M$ and $M$ is orientable. (This solution lacks detail, like how to extend along a curve, what if joining $\tau$ and $\sigma$ isn’t smooth, and explicitly showing $N$ is continuous.)
14. Let \((X, Y)\) be stereographic coordinates on \(S^2 - (0, 0, 1)\) using the north polar projection. Let \((X', Y')\) be stereographic coordinates on \(S^2 - (0, 0, -1)\) using the south polar projection. Compute \(\frac{\partial}{\partial X}, \frac{\partial}{\partial X'}\) and \(\frac{\partial}{\partial Y}, \frac{\partial}{\partial Y'}\).

Solution: The coordinate change \(F\) from north to south is given by \((X, Y) \rightarrow (X', Y') = \frac{1}{X^2 + Y^2}(X, Y)\). Compute

\[
TF = \frac{1}{(X^2 + Y^2)^2} \begin{pmatrix} Y'^2 - X'^2 & -2X'Y' \\ -2X'Y' & X'^2 - Y'^2 \end{pmatrix} = \begin{pmatrix} Y'^2 - X'^2 & -2X'Y' \\ -2X'Y' & X'^2 - Y'^2 \end{pmatrix}.
\]

Then

\[
\frac{\partial}{\partial X} = (Y'^2 - X'^2) \frac{\partial}{\partial X'} - 2X'Y' \frac{\partial}{\partial Y'} \\
\frac{\partial}{\partial Y} = -2X'Y' \frac{\partial}{\partial X'} + (X'^2 - Y'^2) \frac{\partial}{\partial Y'}
\]

and so

\[
\begin{bmatrix} \frac{\partial}{\partial X N}, \frac{\partial}{\partial X S} \end{bmatrix} = 2X' \frac{\partial}{\partial X'} + 2Y' \frac{\partial}{\partial Y'}
\]

\[
\begin{bmatrix} \frac{\partial}{\partial X N}, \frac{\partial}{\partial Y S} \end{bmatrix} = -2Y' \frac{\partial}{\partial X'} + 2X' \frac{\partial}{\partial Y'}
\]

The nicest expression for this result is in spherical coordinates, where

\[
(\theta, \phi) \rightarrow (\cos \theta \cos \phi, \sin \theta \cos \phi, \sin \phi) = (x, y, z) \rightarrow \frac{\cos \phi}{1 + \sin \phi} (\cos \theta, \sin \theta) = (X, Y).
\]

From this, we find

\[
\frac{\partial}{\partial \theta} = -Y' \frac{\partial}{\partial X'} + X' \frac{\partial}{\partial Y'}
\]

\[
\cos \phi \frac{\partial}{\partial \phi} = -X' \frac{\partial}{\partial X'} - Y' \frac{\partial}{\partial Y'}
\]

so that

\[
\begin{bmatrix} \frac{\partial}{\partial X N}, \frac{\partial}{\partial X S} \end{bmatrix} = -2 \cos \phi \frac{\partial}{\partial \phi}
\]

\[
\begin{bmatrix} \frac{\partial}{\partial X N}, \frac{\partial}{\partial Y S} \end{bmatrix} = 2 \frac{\partial}{\partial \theta}
\]

15. The Whitney Embedding Theorem says that any \(m\)-manifold embeds into \(\mathbb{R}^{2m}\). Give one example of an \(m\) manifold that does not embed into \(\mathbb{R}^{2m-1}\).
Solution: When $m = 1$, the circle $S^1$ does not embed into $\mathbb{R}$. Suppose $f : S^1 \to \mathbb{R}$ is an embedding. Since $S^1$ is compact, there is $\theta \in S^1$ such that $f(\theta)$ is the maximum value of $f$. Since $f$ is an embedding, $f$ is a local diffeomorphism, and therefore takes a neighborhood of $\theta$ to a neighborhood of $f(\theta)$, contradicting the maximality of $f(\theta)$. So no such embedding can exists. In fact, there is not even a continuous injective map $S^1 \to \mathbb{R}$. 