Questions get two ratings: A number which is relevance to the course material, a measure of how much I expect you to be prepared to do such a problem on the exam. 3 means ‘of course you know this information’, 1 means ‘you probably need to check something in the book for this one’. Given that you know the material, the starred problems are harder.


(3) 1. Show that a connected manifold is path connected.

(2) 2. Let $D$ be a derivation on $C^\infty(M)$. Suppose $f, g \in C^\infty(M)$, and that $g$ is never 0. Prove the quotient rule:

$$D \left( \frac{f}{g} \right) = \frac{gDf - fDg}{g^2}$$

(3) 3. Given a sequence of open sets $\{U_i\}_{i=1}^\infty$ with $\bar{U}_n \subset U_{n+1}$ for all $n$, and with $\cup_{i=1}^\infty U_n = M$. Say that a sequence $x_1, x_2, \ldots$ leaves all $U$ if for any $n$ there is $N$ so that $x_i \notin U_n$ for $i > N$.

Show that there is a smooth function $f : M \to R$ so that $\lim_{i \to \infty} f(x_i) = +\infty$ for any sequence $\{x_i\}_{i=1}^\infty$ which leaves all $U$.

(3) 4. Which of these homeomorphisms are diffeomorphisms from $\mathbb{R}^2 \to \mathbb{R}^2$?

(a) $(x, y) \to (x^3, y^3)$

(b) $(x, y) \to (x^3 + x, y^3 + y)$

(c) $(x, y) \to (x \cos(x^2 + y^3) − y \sin(x^2 + y^2), x \sin(x^2 + y^2) + y \cos(x^2 + y^2))$

(**2) 5. Let $M(2)$ denote the space of $2 \times 2$ matrices with real entries. Let $N = \{A \in M(2) | A \neq 0, \det(A) = 0\}$. Show that $N$ is a manifold.

(3) 6. For a smooth map of manifolds $f : M \to N$, say that $f$ is self-transverse if for all $x, y \in M$ there are neighborhoods $x \in U$, $y \in V$ so that $f|_U \pitchfork f|_V$.

(a) Give an example of $M, N$ and $f : M \to N$ which is not self-transverse.

(b) Give an example of $M, N$ and $f : M \to N$ which is self-transverse and not injective.

(c) Suppose $f : M \to N$ is a self-transverse immersion. Show $K = \{x \in M | \exists x' \in M$ with $f(x) = f(x')\}$ is a regular submanifold of $M$.

Except that part (c) is false! (*) Give an example to show part (c) is false.

(*2) 7. Let $M$ be a regular submanifold of $N$, and let $X$ be a vector field on $M$. Show there is a vector field $\tilde{X}$ on $N$ with $\tilde{X}|_M = X$.

(2) 8. Show that the set of closed disks in $\mathbb{R}^2$ which don’t contain the origin is a manifold, and show it is diffeomorphic to $S^1 \times \mathbb{R}^2$.

(1) 9. Let $\sigma$ be a curve (embedded 1-manifold) in $\mathbb{R}^3$, and let $\sigma_a$ be the rescaled image of $\sigma$ under the map $(x, y, z) \to (ax, ay, az)$, for some $a > 0$. For $p \in \sigma$, compute the curvature of $\sigma_a$ at $ap$ in terms of $a$ and the curvature of $\sigma$ at $p$.

(2) 10. Suppose $M$ is an embedded surface in $\mathbb{R}^3$, and let $N$ be the rescaled image of $M$ under the map $(x, y, z) \to (ax, ay, az)$, for some $a > 0$. Compute the Gauss curvature $K_N(ap)$ of $N$ at $ap$ in terms of $a$ and the Gauss curvature $K_M(p)$ of $M$ at $p$. 

11. Let \( c = c(s) \) be a unit speed curve in \( \mathbb{R}^3 \), and suppose the Frenet frame \( T, N, B \) is defined for all \( s \). Define \( f : \mathbb{R}^2 \to \mathbb{R}^3 \) by \( f(s, t) = c(s) + tN(s) \). Notice that for fixed \( s \), \( f(s, t) \) is the normal line to the curve at \( c(s) \), and for fixed \( t \), \( f(s, t) \) is a curve ‘parallel’ to \( c \) at distance \( t \).

Find all points where \( f \) fails to be an immersion.

In the case where \( c \) is a planar curve, \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) and these points are the critical values of \( f \).

12. Let \( M(2) \) denote the vector space of \( 2 \times 2 \) matrices. Since \( M(2) \) is a vector space, the tangent space to \( M(2) \) at the identity is naturally identified with \( M(2) \). Let \( SL(2) \subset M(2) \) be the set of matrices of with determinant 1.

(a) Show that \( SL(2) \) is a manifold.

(b) What is \( \dim SL(2) \)?

(c) * Show that the tangent space at the identity, \( T_I SL(2) \), is exactly the space of traceless matrices \( \{ A \in M(2) | tr(A) = 0 \} \).

Bonus: Do this problem for \( n \times n \) matrices instead of \( 2 \times 2 \).

13. Suppose \( M \subset \mathbb{R}^3 \) is a surface, and assume that for any closed curve \( C : S^1 \to M \) there is a continuous unit normal field to \( M \) defined along \( C \). Show that \( M \) is orientable.

14. Let \( (X_N, Y_N) \) be stereographic coordinates on \( S^2 \) using the north polar projection. Let \( (X_S, Y_S) \) be stereographic coordinates on \( S^2 \) using the south polar projection. Compute \( \left[ \frac{\partial}{\partial X_N} \cdot \frac{\partial}{\partial X_S} \right] \) and \( \left[ \frac{\partial}{\partial X_N} \cdot \frac{\partial}{\partial Y_S} \right] \).

15. The Whitney Embedding Theorem says that any \( m \)-manifold embeds into \( \mathbb{R}^{2m} \). Give one example of an \( m \) manifold that does not embed into \( \mathbb{R}^{2m-1} \).