• Boothby pg. 183 #5: Determine the subset of $\mathbb{R}^2$ on which $\sigma_1 = x \, dx + y \, dy$ and $\sigma_2 = y \, dx + x \, dy$ are linearly independent and find a frame field dual to $\sigma_1, \sigma_2$ on this set.

**Solution:** They are independent when \[
\begin{pmatrix} x & y \\ y & x \end{pmatrix}
\] is non-singular. The determinant is $x^2 - y^2$, so they are linearly independent on $\{(x,y) | x^2 - y^2 \neq 0\}$. The dual frame field is described by the inverse matrix

\[
\frac{1}{x^2 - y^2} \begin{pmatrix} x & -y \\ -y & x \end{pmatrix}
\]

and given by $X_1 = (x^2 - y^2)^{-1} \left( x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right)$ and $X_2 = (x^2 - y^2)^{-1} \left( -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right)$.

• Boothby pg. 183 #6: Show that the restriction of $\sigma = x \, dy - y \, dx + z \, dw - w \, dz$ from $\mathbb{R}^4$ to the sphere $S^3$ is never zero on $S^3$.

**Solution:** Let $V = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} + z \frac{\partial}{\partial w} - w \frac{\partial}{\partial z}$. $V$ is a vector field on $S^3$, since it is perpendicular to the radial field $R = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + w \frac{\partial}{\partial w}$. Then compute $\sigma(V) = x^2 + y^2 + z^2 + w^2 = 1$, so $\sigma$ never vanishes on $S^3$.

• Lee Ch 2 Problem 13 (Hessian). Remark: When $df = 0$, $f$ has a critical point, the 'second derivative test' determines the behavior near the critical point using the eigenvalues of the Hessian. For example, in $\mathbb{R}^2$, try the functions $x^2 + y^2$, $x^2 - y^2$, $-x^2 - y^2$, and $xy$ at the origin.

**Solution:** Since $df_p = 0$, $X_p Y f - Y_p X f = [X,Y]_p f = d f_p ([X,Y]_p) = 0$. It is apparent that $H(v,w) = X_p (Y f)$ only depends on $v = X_p$, and not the extension vector field $X$, and is linear in $v = X_p$. Since $X_p (Y f) = Y_p (X f)$, the Hessian $H(v,w)$ depends only on $w$ and not the extension to $Y$, and is linear in $w$.

Suppose $df_p \neq 0$. The claim is that $H(v,w) = X_p (Y f)$ is not well defined - it depends on the choice of $Y$. Choose $w \in T_p M$ so that $df_p (w) = 1$. Let $Y$ be a vector field on $M$ with $Y_p = w$. Let $v$ and $X$ be arbitrary with $X_p = v$. Let $g$ be any smooth function on $M$ with $g(p) = 1$. Then $g Y$ is another extension of $w$, and compute:

$X_p (g Y f) = (X_p g) (Y_p f) + g (p) X_p Y f = X_p g (df (Y_p)) + X_p Y f = X_p g + X_p Y f$.

So, choosing a $g$ with $X_p g \neq 0$, we see that $H(v,w)$ as defined is dependent on the extension of $w$ to $Y$.

• Lee Ch 2 Problem 17 (Test for coordinate charts).
Solution: Since $df_1(p), \ldots, df_N(p)$ spans $T^*_p(M)$, we can discard some of the $f_i$ and renum-
ber them so that $df_1(p), \ldots, df_m(p)$ form a basis for $T^*_p(M)$. Let $x_1, \ldots, x_m$ be coordi-
nates in a neighborhood of $p$, so $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}$ is a basis for $T_p(M)$. Then the matrix
\[ J = \left( df_i \left( \frac{\partial}{\partial x_j} \right) \right) = \left( \frac{\partial f_i}{\partial x_j} \right) \]
is nonsingular, so by the inverse function theorem, there is a neighborhood $V$ of $p$ where $f = (f_1, \ldots, f_m)$ is a diffeomorphism, i.e. a coordinate system on $V$. 