1. Let $M$ be the configuration space of two distinct points on the circle. Identify this with a well known 2-manifold.

2. Identify $\mathbb{R}P^1$ with a well known 1-manifold and give a diffeomorphism.

3. (Lee Exercise 1.46) Show that $\mathbb{C}P^1$ is diffeomorphic to $S^2$.

4. Suppose $M$ is a smooth manifold and $f : M \to \mathbb{R}$ is continuous. Prove for any $\epsilon > 0$ there is a $C^\infty$ function $g : M \to \mathbb{R}$ such that $|f(x) - g(x)| < \epsilon$ for all $x \in M$.

Hint: Use the Stone-Weierstrauss theorem: If $K \subset \mathbb{R}^n$ is compact and $f : K \to \mathbb{R}$ is continuous, then for all $\epsilon > 0$ there is a polynomial $g$ such that $|f(x) - g(x)| < \epsilon$ for all $x \in K$.

**Solution:** Let $\{U_\alpha\}_{\alpha \in A}$ be a locally finite open cover of $M$ so that each $U_\alpha$ has compact closure contained in a chart $(V_\alpha, \psi_\alpha)$. To get such a cover, take a chart $(V_p, \psi_p)$ for each $p \in M$, choose $B_p \subset V_p$ an open neighborhood of $p$ with compact closure $B_p \subset V_p$, and then apply paracompactness to the open cover $\{B_p\}_{p \in M}$.

Let $\{\varphi_\alpha\}_{\alpha \in A}$ be a partition of unity subordinate to $\{U_\alpha\}$.

Fix $\epsilon > 0$ and any $\alpha \in A$. Now $f \circ \psi_\alpha^{-1} : V_\alpha \to \mathbb{R}$ is continuous, so Stone-Weierstrauss gives a smooth function (polynomial, actually) $f_\alpha : V_\alpha \to \mathbb{R}$ so that $|f \circ \psi_\alpha^{-1}(x) - f_\alpha(x)| < \epsilon$ for all $x \in \psi(U_\alpha)$.

Let $g_\alpha : M \to \mathbb{R}$ be $g_\alpha(p) = \varphi_\alpha(p) \cdot f_\alpha(\psi_\alpha(p))$ for $p \in V_\alpha$, and $g_\alpha(p) = 0$ otherwise. The function $g_\alpha$ is smooth, since it is smooth on $V_\alpha$ and any $p \in M - V_\alpha$ has a neighborhood on which $g_\alpha$ is identically zero. Also, the support of $g_\alpha$ is contained in $U_\alpha$.

Define the smooth function $g(p) = \sum_{\alpha \in A} g_\alpha(p)$, which is a finite sum for any $p \in M$ by the local finiteness of $\{U_\alpha\}$.

Finally, for any $p$,

$$|f(p) - g(p)| = |f - \sum_{\alpha} g_\alpha(p)|$$

$$= \left| \sum_{\alpha} \varphi_\alpha(p)f(p) - \sum_{\alpha} \varphi_\alpha(p) \cdot f_\alpha(\psi_\alpha(p)) \right|$$

$$\leq \sum_{\alpha} \varphi_\alpha(p) |f(p) - f_\alpha(\psi_\alpha(p))|$$

$$< \sum_{\alpha} \varphi_\alpha(p) \epsilon$$

$$= \epsilon$$

5. Let $\mathbb{T}^2$ be the 2-torus, given as $\mathbb{T}^2 = \{(e^{i\theta}, e^{i\phi}) \in \mathbb{C}^2 | \theta, \phi \in \mathbb{R}\}$. For real numbers $\alpha, \beta$, there is an action of $\mathbb{R}$ on $\mathbb{T}^2$ given by:

$$t \cdot (e^{i\theta}, e^{i\phi}) = (e^{i(\theta + t\alpha)}, e^{i(\phi + t\beta)}).$$
For which $\alpha, \beta$ is the quotient space $C = \mathbb{R} \setminus \mathbb{T}^2$ Hausdorff?
(Bonus: When $C$ is Hausdorff, what is $C$?)

6. (Lee Ch 1 Exercise 14) The definition of a Lens space.