1. Lee Exercise 1.118. Given smooth \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \), show the graph of \( f \) is a regular submanifold of \( \mathbb{R}^m \times \mathbb{R}^n \).

**Solution:** Define \( \varphi : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{R}^n \) by \( \varphi(x, y) = (x, y - f(x)) \). Clearly, \( \varphi(x, y) = (x, 0) \) if and only if \( (x, y) \) is in the graph of \( f \), i.e. \( f(x) = y \). Next, if \( \varphi(x', y') = \varphi(x, y) \) then \( x = x' \) and so \( y - f(x) = y' - f(x) \), and so \( y = y' \), which shows \( \varphi \) is injective. Finally,

\[
D\varphi = \begin{pmatrix} I & 0 \\ -Df & I \end{pmatrix}
\]

which is nonsingular, so \( \varphi \) is a diffeomorphism. Thus \( \varphi \) is a single-slice chart for the graph of \( f \), making graph \( f \) a regular submanifold of \( \mathbb{R}^m \times \mathbb{R}^n \).

2. Lee Exercise 1.119. The main point of this exercise is to show that every regular submanifold \( M \) of \( \mathbb{R}^n \) is locally the graph of a function.

**Solution:** Note this problem requires the Implicit Mapping Theorem. See Hebda’s notes or Appendix C. It’s not hard to prove, given the Inverse Function Theorem.

Call the standard coordinates on \( \mathbb{R}^n (u_1, \ldots, u_n) \). For \( p \in M \), choose single slice coordinates \( x_1, \ldots, x_n \) on a neighborhood \( U \) of \( p \), so that \( M \cap U = \{(x_1, \ldots, x_k, 0, \ldots, 0) \} \).

Consider each \( x \) coordinate as a function \( x_i = x_i(u_1, \ldots, u_n) \). The matrix \( \left( \frac{\partial x_i}{\partial u_j} \right)_{i,j=1 \ldots n} \) is invertible (it’s the derivative of the change of coordinates function), so the bottom \( n-k \) rows form a rank \( n \) matrix \( \left( \frac{\partial x_i}{\partial u_j} \right)_{i=k+1 \ldots n,j=1 \ldots n} \). This matrix has \( n-k \) independent columns, and by renumbering the \( u_i \) we may assume that the \( n-k \times n-k \) matrix \( \left( \frac{\partial x_i}{\partial u_j} \right)_{i=k+1 \ldots n,j=k+1 \ldots n} \) is invertible. The coordinate plane \( P \) we’re looking for has now been chosen as the plane spanned by \( u_1, \ldots, u_k \), after the renumbering.

Define \( f : U \rightarrow \mathbb{R}^{n-k} \) by \( f(u_1, \ldots, u_n) = (x_{n-k+1}, \ldots, x_n) \). Then \( M \cap U \) is the zero set of \( f \), i.e. \( M \cap U = f^{-1}(0) \).

By the Implicit Mapping Theorem, there is a smooth function \( g : \mathbb{R}^k \rightarrow \mathbb{R}^{n-k} \) so that \( f(u_1, \ldots, u_k, g(u_1, \ldots, u_k)) = 0 \) if and only if \( (u_{k+1}, \ldots, u_n) = g(u_1, \ldots, u_k) \). Then \( (u_1, \ldots, u_k) \rightarrow (u_1, \ldots, u_k, g(u_1, \ldots, u_k)) \) gives a smooth parameterization of \( M \cap U \), showing that \( M \) is locally the graph of a function over the coordinate plane \( P \), and so projection to \( P \) gives a coordinate chart on \( M \).

3. For a real number \( a \), define \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) by \( f(x, y) = x^3 - 3ax - y^2 \). Find all values of \( b \) so that \( f^{-1}(b) \) is a manifold. Graph \( f^{-1}(b) \) for a variety of \( a \) and \( b \), including the critical values of \( b \). These manifolds are called elliptic curves.
Solution: \( Df \) either has rank 1 or is identically zero, and if it has rank 1, then \( f^{-1}(b) \) is a manifold. Solve \( Df = \begin{pmatrix} 3x^2 - 3a \\ -2y \end{pmatrix} = 0 \). First, \( y = 0 \), and we must have \( a \geq 0 \) and then \( x = \pm \sqrt{a} \). Then \( f(\pm \sqrt{a}, 0) = \pm a^{3/2} \mp 3a^{3/2} = \pm 2a^{3/2} \). So the singular values for a given \( a > 0 \) are \( \pm 2a^{3/2} \).

At \( b = -2a^{3/2} \), the surface \( f(x,y) \) has a saddle point, causing a crossing, and at \( b = 2a^{3/2} \) there is a local maximum, causing an isolated point.

When \( a = 0 \), there is a single critical point \( b = 0 \) which is a cusp.

4. Lee Ch 3 Problem 28. For a homogeneous polynomial \( p \) of \( n \) variables, show \( p^{-1}(c) \) is a submanifold of \( \mathbb{R}^n \) for all \( c \neq 0 \).

Solution: For \( x \in p^{-1}(c) \), define a curve \( \sigma(t) = (1 + t)x \). Then \( p \circ \sigma : \mathbb{R} \to \mathbb{R} \). I will show \( p \circ \sigma \) has rank 1 at \( t = 0 \) and therefore \( p \) has rank 1 at \( x \).

\[
T_0(p \circ \sigma) = \frac{d}{dt} p(\sigma(t)) \bigg|_{t=0} = \frac{d}{dt} (1 + t)^m p(x) \bigg|_{t=0} = mc \neq 0.
\]

Since \( p \) has rank 1 at \( x \) for all \( x \in p^{-1}(c) \), \( c \) is a regular value for \( p \) and therefore \( p^{-1}(c) \) is a manifold.

5. Lee Ch 3 Problem 3. Show if \( M \) is compact and \( N \) is connected, then a submersion \( f : M \to N \) must be surjective.

Solution: We’ll show \( f(M) \) is both open and closed in \( N \). First, if \( y \in f(M) \) then \( y = f(x) \) for some \( x \). In local coordinates near \( x \) and \( y \), \( f \) has the form \( f(x_1, \ldots, x_m) = (x_1, \ldots, x_n) \), so an open set around \( x \) maps to an open set around \( y \), and \( f(M) \) is open. Now suppose
there is a sequence $y_1, y_2, \ldots$ converging to $y$ in $N$, with $y_i = f(x_i)$. Since $M$ is compact, by passing to a subsequence $y_i \to x \in M$. Since $f$ is continuous, $f(y_i) \to f(x)$, so $y = f(x)$ and $f(M)$ is closed.

Since $N$ is connected, $f(M) = N$ and $f$ is surjective.

6. Define the map $H : \mathbb{R}^4 \to \mathbb{R}^3$ by

$$H(x, y, z, w) = (2(xy + zw), 2(xw - yz), x^2 + z^2 - y^2 - w^2).$$

Check that restricting $H$ to $S^3 \subset \mathbb{R}^4$ defines a map from $S^3$ to $S^2$, the Hopf map. Show that the Hopf map is a submersion. What are the fibers of the Hopf map (i.e. what manifold is $f^{-1}(q)$ for $q \in S^2$)?