On this exam, $M$ will always be a smooth $m$-dimensional manifold.

1. Give an example of a topological space $X$ which is locally Euclidean but not a manifold. (Define the topology on $X$ and show both claims)

2. With $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$, let $C \subset S^2$ consist of the three circles $z = \frac{1}{2}$, $z = 0$, $z = -\frac{1}{2}$. Accurately sketch the image of $C$ under stereographic projection from the north pole $(0,0,1)$.

3. Given smooth functions $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$ and distinct points $p, q \in M$, show that there is a smooth function $h : M \to \mathbb{R}$ so that $h \equiv f$ in a neighborhood of $p$, and $h \equiv g$ in a neighborhood of $q$.

4. Let $\Delta \subset M \times M$ be the diagonal, $\Delta = \{(p, p) | p \in M\}$. Describe charts on $\Delta$ that make $\Delta$ an $m$-manifold diffeomorphic to $M$. (You don’t need to prove anything, just define the charts).

5. Let $f : M \to \mathbb{R}$ be a smooth function, and suppose $f$ takes its maximum value at $p \in M$.
   For any $X_p \in T_pM$, show that $X_p f = 0$.

6. Show that
   
   $$f([x : y]) = \frac{xy}{x^2 + y^2}$$
   
   is well defined as a function $f : \mathbb{RP}^1 \to \mathbb{R}$, where $[x : y]$ are homogenous coordinates on projective space.
   Find the maximum of $f$ on $\mathbb{RP}^1$ (you may use problem 5).